Example: (pg 242)

$$\frac{dX}{dt} = AX \quad \text{where } A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

Eigenvalues and eigenvectors:

$$\det(A - \lambda I) = \det \begin{bmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix} = (6-\lambda)(4-\lambda) + 5 = 0$$

get complex values of λ.

$$\begin{array}{r} {}^{3}29 \\ 4 \\ \hline 116 \end{array}$$

$$\lambda^2 - 10\lambda + 24 + 5 = \lambda^2 - 10\lambda + 29 = 0$$

$a = 1 \quad b = -10 \quad c = 29$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 - 116}}{2} = \frac{10 \pm i\sqrt{16}}{2} = \frac{5 \pm 4i}{2} = 5 \pm 2i$$

If the eigenvalues are complex the eigenvectors need to be complex as well (since the matrix A is real).

$$AK_1 = (5 - 2i)K_1 \qquad AK_2 = (5 + 2i)K_2$$

$\lambda_1 = 5 - 4i$

$$\begin{bmatrix} 6 - (5 - 2i) & -1 \\ 5 & 4 - (5 - 2i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 + 2i & -1 \\ 5 & -1 + 2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$a(1 + 2i) - b = 0 \qquad b = a(1 + 2i)$$

take $a = 1$ then $b = 1 + 4i$

$$K_1 = \begin{bmatrix} 1 \\ 1 + 2i \end{bmatrix}$$

To make sure there is no error check this also satisfies the other equation

$$5a + (-1+2i)b = 0$$

Check

$$5\cdot 1 + (-1+2i)(1+2i) = 5 + (2i-1)(2i+1)$$

$$= 5 + 4i^2 - 1 + \cancel{2i} - \cancel{2i} = 5 - 4 - 1 = 0 \quad \boxed{\checkmark}$$

the other eigenvalue

$$\overline{AK_1} = \overline{(5-2i)K_1} \qquad \text{then} \qquad A\overline{K_1} = (5+2i)\overline{K_1}$$

since A is a real matrix

compare with $AK_2 = (5+2i)K_2$

Thus

$$K_2 = \overline{K_1} = \overline{\begin{bmatrix} 1 \\ 1+2i \end{bmatrix}} = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

General solution:

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} e^{(5-2i)t} + c_2 \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{(5+2i)t}$$

Since $e^{x+i\theta} = e^x e^{i\theta} = e^x(\cos\theta + i\sin\theta)$

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} e^{5t}(\cos 2t - i\sin 2t) + c_2 \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{5t}(\cos 2t + i\sin 2t)$$

$$= c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} i\right) e^{5t}(\cos 2t - i\sin 2t) + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} i\right) e^{5t}(\cos 2t + i\sin 2t)$$

$$= c_1\begin{bmatrix}1\\1\end{bmatrix}e^{5t}\cos 2t + c_1\begin{bmatrix}0\\2\end{bmatrix}e^{5t}\sin 2t - ic_1\begin{bmatrix}1\\1\end{bmatrix}e^{5t}\sin 2t + ic_1\begin{bmatrix}0\\2\end{bmatrix}e^{5t}\cos 2t$$

$$+ c_2\begin{bmatrix}1\\1\end{bmatrix}e^{5t}\cos 2t + c_2\begin{bmatrix}0\\2\end{bmatrix}e^{5t}\sin 2t + ic_2\begin{bmatrix}1\\1\end{bmatrix}e^{5t}\sin 2t - ic_2\begin{bmatrix}0\\2\end{bmatrix}e^{5t}\cos 2t$$

$$= e^{5t}\left(c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} + ic_1\begin{bmatrix}0\\2\end{bmatrix} - ic_2\begin{bmatrix}0\\2\end{bmatrix}\right)\cos 2t$$

$$+ e^{5t}\left(c_1\begin{bmatrix}0\\2\end{bmatrix} + c_2\begin{bmatrix}0\\2\end{bmatrix} - ic_1\begin{bmatrix}1\\1\end{bmatrix} + ic_2\begin{bmatrix}1\\1\end{bmatrix}\right)\sin 2t$$

$$= e^{5t}\left(\underbrace{(c_1+c_2)}_{\alpha}\begin{bmatrix}1\\1\end{bmatrix} + \underbrace{i(c_1-c_2)}_{\beta}\begin{bmatrix}0\\2\end{bmatrix}\right)\cos 2t$$

$$+ e^{5t}\left(\underbrace{(c_1+c_2)}_{\alpha}\begin{bmatrix}0\\2\end{bmatrix} - \underbrace{i(c_1-c_2)}_{-\beta}\begin{bmatrix}1\\1\end{bmatrix}\right)\sin 2t$$

$$= e^{5t}\left[\left(\alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}0\\2\end{bmatrix}\right)\cos 2t + \left(\alpha\begin{bmatrix}0\\2\end{bmatrix} - \beta\begin{bmatrix}1\\1\end{bmatrix}\right)\sin 2t\right]$$

here α, β are my constants $\alpha = c_1 + c_2$, $\beta = i(c_1 - c_2)$

$$= e^{5t}\left[\alpha\left(\begin{bmatrix}1\\1\end{bmatrix}\cos 2t + \begin{bmatrix}0\\2\end{bmatrix}\sin 2t\right) + \beta\left(\begin{bmatrix}0\\2\end{bmatrix}\cos 2t - \begin{bmatrix}1\\1\end{bmatrix}\sin 2t\right)\right]$$

$$= e^{5t}\left(\alpha\begin{bmatrix}\cos 2t\\ \cos 2t + 2\sin 2t\end{bmatrix} + \beta\begin{bmatrix}-\sin 2t\\ 2\cos 2t - \sin 2t\end{bmatrix}\right)$$

$$X_1(t) = e^{5t}\begin{bmatrix}\cos 2t\\ \cos 2t + 2\sin 2t\end{bmatrix} \qquad X_2(t) = e^{5t}\begin{bmatrix}-\sin 2t\\ 2\cos 2t - \sin 2t\end{bmatrix}$$