Complex Eigenvalues

complex conjugate pairs

$\lambda_1 = 5 - 2i \qquad K_1 = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} \qquad X_1(t) = e^{5t}\begin{bmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix}$

$\lambda_2 = 5 + 2i \qquad K_2 = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} \qquad X_2(t) = e^{5t}\begin{bmatrix} -\sin 2t \\ 2\cos 2t - \sin 2t \end{bmatrix}$

$\lambda_2 = \overline{\lambda_1}$ and $K_2 = \overline{K_1}$

before

$\alpha = 5 \qquad \beta = -2$

General situation

$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\alpha, \beta \in \mathbb{R}$

$B_1, B_2 \in \mathbb{R}^2$

$\lambda_1 = \alpha + \beta i \qquad K_1 = B_1 + i B_2$

$\lambda_2 = \alpha - \beta i \qquad K_2 = B_1 - i B_2$

$$X_1(t) = e^{5t}\begin{bmatrix} \cos(-2t) \\ \cos(-2t) - 2\sin(-2t) \end{bmatrix} = e^{\alpha t}\left[B_1 \cos\beta t - B_2 \sin\beta t\right]$$

$$X_2(t) = e^{5t}\begin{bmatrix} +\sin(-2t) \\ 2\cos(-2t) + \sin(-2t) \end{bmatrix} = e^{\alpha t}\left[B_2 \cos\beta t + B_1 \sin\beta t\right]$$

THEOREM 8.2.3 Real Solutions Corresponding to a Complex Eigenvalue

Let $\lambda_1 = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $\mathbf{A}$ in the homogeneous system (2) and let $\mathbf{B}_1$ and $\mathbf{B}_2$ denote the column vectors define in (22). Then

$$\begin{aligned} \mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t]e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t]e^{\alpha t} \end{aligned} \tag{23}$$

are linearly independent solutions of (2) on $(-\infty, \infty)$.

Inhomogeneous Systems. §8.3

- Guess solution and solve for the constants
- Variation of Parameters

$$\frac{dX}{dt} = AX + F(t)$$

← time dependency...

First solve homogeneous problem:

$$\frac{dX_h}{dt} = AX_h \qquad \text{note} \quad \frac{dX_1}{dt} = AX_1, \quad \frac{dX_2}{dt} = AX_2, \quad \dots$$

The general solution $X_h(t) = c_1X_1(t) + c_2X_2(t) + \dots + c_nX_n(t)$

Fundamental solution matrix

x_{ij}: first index is row, second index is column

$$\Phi(t) = \left[X_1(t) \,\vdots\, X_2(t) \,\vdots\, \cdots \,\vdots\, X_n(t) \right] = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

Then.

$$X_h(t) = \Phi(t)\,C \qquad \text{where} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$X_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix} \qquad X_2(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix} \quad \dots$$

↑ second index is the column..

$$X_h'(t) = \frac{d}{dt}\left(\Phi(t)\,C\right) = \left(\frac{d}{dt}\left[X_1(t) \,\vdots\, X_2(t) \,\vdots\, \cdots \,\vdots\, X_n(t) \right]\right) C$$

$$= \left[\frac{d}{dt} X_1(t) \;\vdots\; \frac{d}{dt} X_2(t) \;\vdots\; \cdots \;\vdots\; \frac{d}{dt} X_n(t) \right] C$$

doesn't depend on A being constant

$$= \left[A X_1(t) \;\vdots\; A X_2(t) \;\vdots\; \cdots \;\vdots\; A X_n(t) \right] C$$

$$= A \left[X_1(t) \;\vdots\; X_2(t) \;\vdots\; \cdots \;\vdots\; X_n(t) \right] C = A \Phi(t) C = A X_h$$

Note, what really happened

$$\frac{d\Phi(t)}{dt} = A \Phi(t)$$

Now use the solution to the homogeneous problem to solve the inhomogeneous problem.

$$X_p(t) = \Phi(t)\, U(t)$$

unknown vector $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$

Idea solve for U..

plug in to the original problem

$A\Phi(t)$

$$\frac{dX}{dt} = AX + F(t)$$

$$\frac{d}{dt}\left(\Phi(t)\, U(t)\right) = \Phi'(t) U(t) + \Phi(t) U'(t) = A \Phi(t) U(t) + F(t)$$

$$= A\Phi(t) U(t) + \Phi(t) U'(t) = A \Phi(t) U(t) + F(t)$$

Thus

$$\Phi(t)U'(t) = F(t)$$

Solve for $U'(t)$

$$U'(t) = [\Phi(t)]^{-1} F(t)$$

$$U(t) = \int [\Phi(t)]^{-1} F(t)\,dt$$

Therefore Variation of Parameters formula...

$$X_p(t) = \Phi(t)U(t) = \Phi(t)\int [\Phi(t)]^{-1} F(t)\,dt$$

General formula works for any F and even when A is not constant, but you need to know $X_1, X_2, \ldots, X_n$ solutions to the homogeneous equation.

In our examples we assume A is constant so that we can find $X_1, \ldots, X_n$ using eigenvalues and eigenvectors.

General solution

$$X(t) = X_h(t) + X_p(t)$$

...

$$= c_1X_1(t) + \cdots + c_nX_n(t) + \Phi(t)\int [\Phi(t)]^{-1} F(t)\,dt$$

Next time examples...