Example: (pg 242)

Homogeneous problem

$$\frac{dX}{dt} = AX \quad \text{where } A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

Consider the more complicated (inhomogeneous) problem

$$\frac{dX}{dt} = AX + F(t) \qquad F(t) = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

variation of parameters...

$$X_p(t) = \Phi(t) \int [\Phi(t)]^{-1} F(t)\, dt$$

$$\Phi(t) = \begin{bmatrix} X_1(t) \,\vdots\, X_2(t) \end{bmatrix} = \begin{bmatrix} e^{5t}\cos 2t & -e^{5t}\sin 2t \\ e^{5t}(\cos 2t + 2\sin 2t) & e^{5t}(2\cos 2t - \sin 2t) \end{bmatrix}$$

Idea, in simple cases guess and solve for constants...

$$\frac{dX}{dt} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} X + \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Guess $X_p(t)$ and solve for constants...

$$X_p = \begin{bmatrix} a \\ b \end{bmatrix}$$

guess a constant vector since F is a constant vector.

Plug in and if we can solve for a, b such that this is a solution, then we're done..

$$\frac{dX_p}{dt} = \frac{d}{dt}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} X_p + \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 6a - b + 7 \\ 5a + 4b + 1 \end{bmatrix}$$

Therefore $\begin{bmatrix} 6a - b + 7 \\ 5a + 4b + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} (6a - b = -7)\,4 \\ 5a + 4b = -1 \end{cases}$$

$$\begin{aligned} 24a - \cancel{4b} &= -28 \\ 5a + \cancel{4b} &= -1 \\ \hline 29a &= -29 \end{aligned}$$

so $a = -1$

Now solve for b

$$5(-1) + 4b = -1$$
$$4b = -1 + 5 = 4$$
$$b = 1$$

$$X_p = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

General solution to the inhomogeneous problem.

So $$X(t) = X_h(t) + X_p(t) = c_1 X_1(t) + c_2 X_2(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X(t) = c_1 e^{5t}\begin{bmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} -\sin 2t \\ 2\cos 2t - \sin 2t \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

From the beginning of linear systems.

$$\frac{dX}{dt} = AX \qquad \text{I'd like to say } X = e^{At} c$$

give meaning to this

e^{At} what is this?

Solutions to the equation were X_1 and X_2. We defined.

$$\Phi(t) = \left[X_1(t) \,\vdots\, X_2(t) \right]$$

$$X(t) = \Phi(t) c = \left[X_1(t) \,\vdots\, X_2(t) \right] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 X_1(t) + c_2 X_2(t)$$

It seems like $\Phi(t)$ and e^{At} should be the same...

Problem... $\Phi(t)$ is not uniquely defined, for example

$\tilde{\Phi}(t) = \left[X_2(t) \,\vdots\, X_1(t) \right]$ works just as well except different constants c.

To make unique we'll require $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\Phi(t) = e^{At}$$

This is natural because $e^{At} = e^{0} = I$

zero matrix.

identity matrix