$$\Phi(t) = \left[X_1(t) \,\vdots\, X_2(t) \right]$$

$X(t) = \Phi(t)\, C$ constants

Suppose $X(0) = X_0$ is the initial condition

How to solve for C

$X(0) = \Phi(0)\, C = X_0$

Since $e^{A0} = I$

$X(t) = e^{At} C$

a matrix...

$C = \Phi(0)^{-1} X_0$

$X(0) = I C = X_0$

$C = X_0$

$X(t) = \underbrace{\Phi(t)\, \Phi(0)^{-1}} X_0$

Want $X(t) = \underbrace{e^{At}} X_0$

so $e^{At} = \Phi(t)\, \Phi(0)^{-1}$...

$$\Phi(t) = \left[X_1(t) \,\vdots\, X_2(t) \,\vdots\, \cdots \,\vdots\, X_n(t) \right]$$

$X_1(t) = K_1 e^{\lambda_1 t}$, $X_2(t) = K_2 e^{\lambda_2 t}$ ... $X_n(t) = K_n e^{\lambda_n t}$

$$\Phi(t) = \left[K_1 e^{\lambda_1 t} \,\vdots\, K_2 e^{\lambda_2 t} \,\vdots\, \cdots \,\vdots\, K_n e^{\lambda_n t} \right]$$

$$= \left[K_1 \,\vdots\, K_2 \,\vdots\, \cdots \,\vdots\, K_n \right] \begin{bmatrix} e^{\lambda_1 t} & 0 & & O \\ & e^{\lambda_2 t} & & \\ O & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

$$\Phi(0) = \left[K_1 \,\vdots\, K_2 \,\vdots\, \cdots \,\vdots\, K_n \right] \begin{bmatrix} e^{\lambda_1 0} & 0 & & \bigcirc \\ & e^{\lambda_2 0} & & \\ \bigcirc & & \ddots & \\ & & & e^{\lambda_n 0} \end{bmatrix}$$

$$= \left[K_1 \,\vdots\, K_2 \,\vdots\, \cdots \,\vdots\, K_n \right] \begin{bmatrix} 1 & 0 & & \bigcirc \\ & 1 & & \\ \bigcirc & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$= \left[K_1 \,\vdots\, K_2 \,\vdots\, \cdots \,\vdots\, K_n \right]$$

$$e^{At} = \Phi(t)\,\Phi(0)^{-1}$$

$$= \left[K_1 \,\vdots\, K_2 \,\vdots\, \cdots \,\vdots\, K_n \right] \begin{bmatrix} e^{\lambda_1 t} & 0 & & \bigcirc \\ & e^{\lambda_2 t} & & \\ \bigcirc & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \left[K_1 \,\vdots\, K_2 \,\vdots\, \cdots \,\vdots\, K_n \right]^{-1}$$

Does this make sense? think about what e^x really is...

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \ldots$$

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \frac{1}{4!}(At)^4 + \ldots$$

$AK_i = \lambda_i K_i$ definition of eigenvalues and eigenvectors.

$$A \underbrace{\left[K_1 \,\vdots\, K_2 \,\vdots\, \cdots \,\vdots\, K_n \right]}_{\Phi(0)} = \left[AK_1 \,\vdots\, AK_2 \,\vdots\, \cdots \,\vdots\, AK_n \right]$$

$$= \left[\lambda_1 K_1 \,\vdots\, \lambda_2 K_2 \,\vdots\, \cdots \,\vdots\, \lambda_n K_n \right]$$

$$A\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}^{-1}}_{U^{-1}}$$

$$A = UDU^{-1}$$

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \frac{1}{4!}(At)^4 + \cdots$$

$$= I + UDU^{-1}t + \frac{1}{2!}(UDU^{-1}t)^2 + \frac{1}{3!}(UDU^{-1}t)^3 + \cdots$$

$$= I + UDU^{-1}t + \frac{1}{2!}(UDU^{-1})^2 t^2 + \frac{1}{3!}(UDU^{-1})^3 t^3 + \cdots$$

$$(UDU^{-1})^2 = UDU^{-1}UDU^{-1} = UD^2U^{-1}$$

$$(UDU^{-1})^3 = UDU^{-1}UDU^{-1}UDU^{-1} = UD^3U^{-1}$$

$$e^{At} = I + UDU^{-1}t + \frac{1}{2!}UD^2U^{-1}t^2 + \frac{1}{3!}UD^3U^{-1}t^3 + \cdots$$

$$= UU^{-1} + UDU^{-1}t + \frac{1}{2!}UD^2U^{-1}t^2 + \frac{1}{3!}UD^3U^{-1}t^3 + \cdots$$

$$= U\left(I + Dt + \frac{1}{2!}D^2t^2 + \frac{1}{3!}D^3t^3 + \cdots\right)U^{-1}$$

$$= U\left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & & \\ & \lambda_2 t & & \\ & & \ddots & \\ & & & \lambda_2 t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 t^2 & & & \\ & \lambda_2^2 t^2 & & \\ & & \ddots & \\ & & & \lambda_2^2 t^2 \end{bmatrix} + \cdots\right)U^{-1}$$

$$= U \begin{bmatrix} 1+\lambda_1 t+\lambda_1^2 \frac{t^2}{2!}+\lambda_1^3\frac{t^3}{3!}+\cdots & & & \\ & 1+\lambda_2 t+\lambda_2^2\frac{t^2}{2!}+\lambda_2^3\frac{t^3}{3!}+\cdots & & \\ & & \ddots & \\ & & & 1+\lambda_n t+\lambda_n^2\frac{t^2}{2!}+\lambda_n^3\frac{t^3}{3!}+\cdots \end{bmatrix} U^{-1}$$

$$= U \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} U^{-1}$$

$$e^{At} = \begin{bmatrix} K_1 & \vdots & K_2 & \vdots & \cdots & \vdots & K_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & & O \\ & e^{\lambda_2 t} & & \\ O & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} K_1 & \vdots & K_2 & \vdots & \cdots & \vdots & K_n \end{bmatrix}^{-1}$$

Solution to $\frac{dX}{dt} = AX$ is $X(t) = e^{At}\underbrace{X(0)}_{\text{initial cond.}}$

1. $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad K_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad K_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_1 = 1 \quad \lambda_2 = 2$

$$e^{At} = \begin{bmatrix} K_1 & \vdots & K_2 & \vdots & \cdots & \vdots & K_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & & O \\ & e^{\lambda_2 t} & & \\ O & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} K_1 & \vdots & K_2 & \vdots & \cdots & \vdots & K_n \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

the inv of identity is also the indentity

9. $\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds.$$

$$X_c = e^{At}C = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} C$$

$$X_p = e^{At}\int_{t_0}^{t} e^{-As}\begin{bmatrix} 3 \\ -1 \end{bmatrix} ds = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\int_{t_0}^{t}\begin{bmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} ds$$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\int_{t_0}^{t}\begin{bmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{bmatrix} ds \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -e^{-t}+e^{-t_0} & 0 \\ 0 & -\frac{1}{2}e^{-t}+\frac{1}{2}e^{-t_0} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} C + \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -e^{-t}+e^{-t_0} & 0 \\ 0 & -\frac{1}{2}e^{-t}+\frac{1}{2}e^{-t_0} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= c_1e^t + c_2e^{2t} + 3\left(e^{-t_0} - e^{-t}\right)e^t + \left(\tfrac{1}{2}e^{-t} - \tfrac{1}{2}e^{-t_0}\right)e^{2t}$$

8.3 example of variation of parameters

Solve the system

$$\mathbf{X}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} \tag{11}$$

on $(-\infty, \infty)$.

SOLUTION We first solve the associated homogeneous syste

$$\mathbf{X}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\mathbf{X}. \tag{12}$$

The characteristic equation of the coefficient matrix i

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda + 2)(\lambda + 5) = 0,$$

so the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -5$. By the usual method we find that the eigenvectors corresponding to λ_1 and λ_2 are, respectively, $\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{K}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. The solution vectors of the homogeneous system (12) are then

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t} = \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix}.$$

The entries in $\mathbf{X}_1$ form the first column of $\mathbf{\Phi}(t)$, and the entries in $\mathbf{X}_2$ form the second column of $\mathbf{\Phi}(t)$. Hence

$\Phi(t) = [X_1 \,\vdots\, X_2]$

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1}(t) = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}.$$

From (9) we obtain the particular solution

$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds.$

e^{At} e^{-As} $F(s)ds$

$$\mathbf{X}_p = \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} dt$$

more next time ...