Math 285 Sample Exam 1 Version A

1. Solve the initial value problem $y' + 3y = e^{-3x}$ with $y(0) = 7$.

Linear equation. Integrating factor

$$\mu = e^{3x}$$

Thus

$$(ye^{3x})' = 1$$

So

$$ye^{3x} = x + C$$

General solution:

$$y = xe^{-3x} + Ce^{-3x}$$

Unique solution. Solve for C.

$$y(0) = C = 7 \quad \text{so} \quad C = 7$$

Answer:

$$y = (x + 7)e^{-3x}.$$

2. Solve $(e^{2y} - y)(\cos x)\dfrac{dy}{dx} = e^y \sin 2x$ with $y(0) = 0$.

seperable.

$$\int (e^{y} - ye^{-y})\,dy = \int \frac{\sin 2x}{\cos x}\,dx = \int \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}\,dx$$

Guessing (or integrating by parts) yields

$$\int ye^{-y}\,dy = (Ay+B)e^{-y}$$

$$\frac{d}{dy}(Ay+B)e^{-y} = (A - B - Ay)e^{-y} = ye^{-y}$$

Thus $A=-1$ and $B=A=-1$.

Thus

$$e^{y} + (y+1)e^{-y} = \int 2\sin x\,dx = -2\cos x + C$$

General solution

$$e^{y} + (y+1)e^{-y} = -2\cos x + C$$

Solve for C to obtain unique solution

$$1 + (0+1)e^{-0} = -2\cos 0 + C \qquad \text{so } C = 4$$

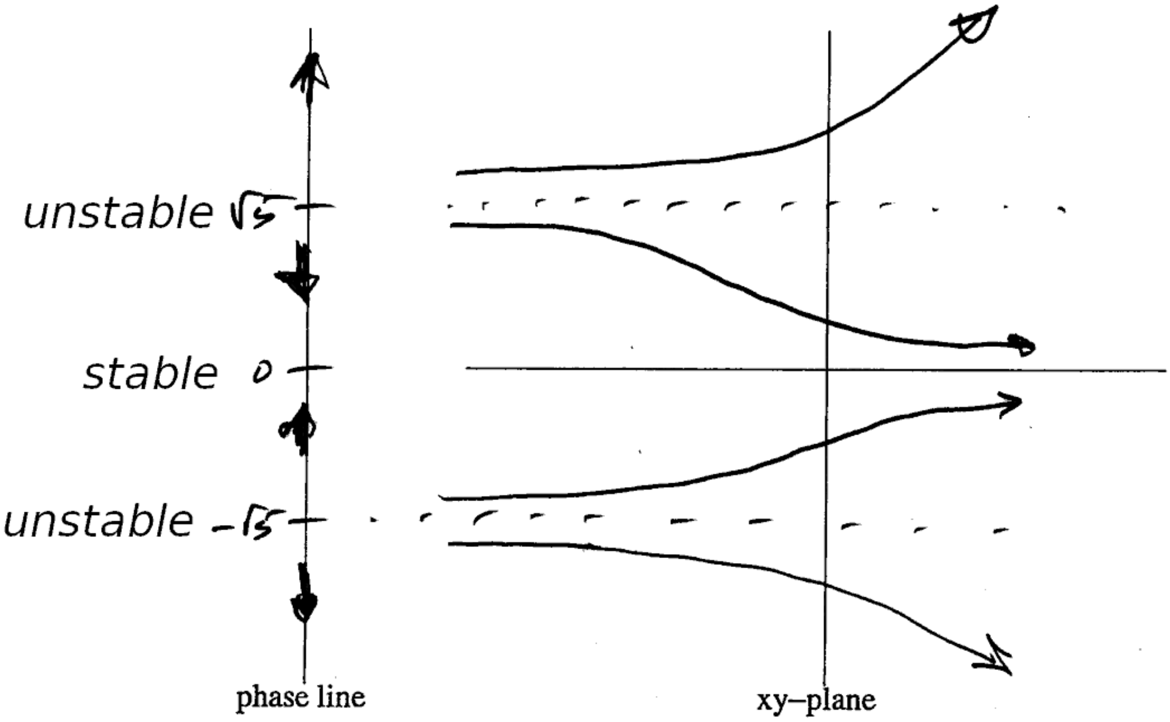
Unique solution

$$e^{y} + (y+1)e^{-y} = -2\cos x + 4.$$

Quiz 1

Math 285 ~~Sample Exam~~ 2 Version B

3. Draw a phase portrait and solution curves for the autonomous first-order ordinary differential equation $y' = y^3 - 5y$ below. Label the stationary points and determine whether they are stable, unstable or semi-stable.



$$y^3 - 5y = y(y^2-5) = y(y-\sqrt{5})(y+\sqrt{5})$$

$$y = 0, -\sqrt{5}, \sqrt{5}$$

4. Show that the ordinary differential equation

$$(y\cos x + 2xe^{y})dx + (\sin x + x^2e^{y} + 2)dy = 0$$

is exact and find the general solution.

$$M_y = \frac{\partial}{\partial y}(y\cos x + 2xe^{y}) = \cos x + 2xe^{y}$$

$$N_x = \frac{\partial}{\partial x}(\sin x + x^2e^{y} + 2) = \cos x + 2xe^{y}$$

Since $M_y = N_x$ then the equation is exact.

Solve, holding y constant

$$\psi = \int(y\cos x + 2xe^{y})\,dx = y\sin x + x^2e^{y} + g(y)$$

$$\psi_y = \sin x + x^2e^{y} + g'(y) = \sin x + x^2e^{y} + 2$$

so $g'(y) = 2$ and $g(y) = 2y + C$.

It follows that the general solution is

$$y\sin x + x^2e^{y} + 2y = C.$$

5. Find the unique solution to $\frac{dy}{dx} = \frac{2x}{1+2y}$ with $y(0) = 1$.

Seperable

$$\int (1+2y)\,dy = \int 2x\,dx$$

Thus

$$y + y^2 = x^2 + C$$ is the general solution.

Solve for C.

$$1 + 1^2 = 0^2 + C \quad \text{so} \quad C = 2.$$

Hence

$$y + y^2 = x^2 + 2.$$

Note since y is given implicity in terms of a quadratic equation there is still a sign indeterminacy in the function y. To make this unique, we solve the quadratic equation.

$$y^2 + y - (x^2+2) = 0$$

with $a = 1$, $b = 1$, $c = -(x^2+2)$ to obtain

$$y = \frac{-1 \pm \sqrt{1 + 4(x^2+2)}}{2} = \frac{-1 \pm \sqrt{4x^2+9}}{2}.$$

Since $y(0)=1$ we select the "plus" solution.

Unique solution:

$$y = \frac{-1 + \sqrt{4x^2+9}}{2}.$$

6. Find the general solution to the differential equation

$$(y^2 + 2xy)dx - x^2dy = 0. \quad \leftarrow \frac{dy}{dx} = \frac{y^2+2xy}{x^2}$$

This equation is homogeneous in x and y so we use the substitution $y = ux$. Thus

$$\frac{dy}{dx} = \frac{du}{dx}\cdot x + u$$

and consequently

$$\frac{du}{dx}\cdot x + u = \frac{y^2+2xy}{x^2} = \frac{u^2x^2 + 2x^2u}{x^2}$$

$$\frac{du}{dx}\cdot x = u^2 + u$$

This is separable.

$$\int \frac{du}{u(u+1)} = \int \frac{dx}{x} = \ln|x| + c$$

By partial fractions

$$\int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right)du = \ln|u| - \ln|u+1|$$

Thus $\ln\left|\frac{u}{u+1}\right| = \ln|x| + C$

or $\left|\frac{u}{u+1}\right| = |x|e^C$ or $\frac{u}{u+1} = Cx$

General solution $\frac{y}{y+x} = Cx$ or $y = Cxy + Cx^2$

$$y = \frac{Cx^2}{1-Cx} \quad \text{or} \quad y = \frac{x^2}{C-x}$$

7. Find the general solution to the differential equation

$$x\frac{dy}{dx} + y = x^2y^2.$$

This is Bernoulli equation with $n=2$. Thus substitute $u = y^{1-n} = y^{-1}$ or $y = \frac{1}{u}$.

By chain rule

$$\frac{dy}{dx} = \frac{d}{dx}\frac{1}{u} = \frac{-1}{u^2}\frac{du}{dx}$$

and the differential equation becomes

$$x\left(\frac{-1}{u^2}\right)\frac{du}{dx} + \frac{1}{u} = \frac{x^2}{u^2}$$

or $$\frac{du}{dx} - \frac{u}{x} = -x$$

This is linear, the integrating factor is

$$\mu = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

so

$$\frac{d}{dx}\left(\frac{u}{x}\right) = -1 \quad \text{so} \quad \frac{u}{x} = -x + C.$$

$$\text{or} \quad \frac{1}{xy} = -x + C \quad \text{or} \quad xy = \frac{1}{C-x}$$

General solution is

$$y = \frac{1}{Cx - x^2}.$$