Math 285: Sample Exam 2 Version A

1. Find the unique solution to the Cauchy-Euler equation

$$x^2y'' + xy' - 4y = 0 \qquad \text{where} \qquad y(1) = 3 \quad \text{and} \quad y'(1) = 2.$$

2. Use algebra and theorems on the last page to find

(i) $\mathcal{L}\{t^2 - e^{-9t} + 5\}(s)$

(ii) $\mathcal{L}^{-1}\left\{\dfrac{2s-6}{s^2+9}\right\}(t)$

3. Use the provided theorems to find

(i) $\mathcal{L}\{e^t \sin 3t\}(s)$.

(ii) $\mathcal{L}\left\{ \int_0^t e^{-\tau} \cos \tau \, d\tau \right\}(s)$.

4. Use the provided theorems to solve the initial-value problem

$$y'' + y = \delta(t - 2\pi) \qquad \text{where} \qquad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

5. Write the system

$$\frac{dX}{dt} = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

without the use of matrices.

6. The matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

has eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \text{and} \qquad K_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$ and $\lambda_3 = 3$. Find the general solution to $dX/dt = AX$.

THEOREM 7.1.1 Transforms of Some Basic Functions

$\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for $a \geq 0$

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

THEOREM 7.2.1 Some Inverse Transforms

$\mathscr{L}^{-1}\{e^{-as}\} = \delta(t-a)$ for $a \geq 0$

(a) $1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \ldots$

(c) $e^{at} = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2 + k^2}\right\}$

(e) $\cos kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2 + k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2 - k^2}\right\}$

(g) $\cosh kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2 - k^2}\right\}$

THEOREM 7.2.2 Transform of a Derivative

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

THEOREM 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s - a).$$

THEOREM 7.3.2 Second Translation Theorem

If $F(s) = \mathscr{L}\{f(t)\}$ and a 0, then

$$\mathscr{L}\{f(t - a)\,\mathscr{U}(t - a)\} = e^{-as}F(s).$$

THEOREM 7.4.1 Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

THEOREM 7.4.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

THEOREM 7.4.3 Transform of a Periodic Function

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$