Math 285: Sample Exam 2 Version A

**1.** Find the unique solution to the Cauchy-Euler equation

$$x^2y'' + xy' - 4y = 0 \qquad \text{where} \qquad y(1) = 3 \quad \text{and} \quad y'(1) = 2.$$

Look for solution of the form $y = x^m$. Plug it in

$y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Thus

$x^2 m(m-1)x^{m-2} + x m x^{m-1} - 4x^m = 0$

$m(m-1) + m - 4 = 0 \qquad m^2 - m + m - 4 = 0$

$m^2 - 4 = 0 \qquad (m-2)(m+2) = 0 \qquad \text{so} \quad m = 2, -2.$

General solution

$y(x) = c_1x^2 + c_2x^{-2} \qquad y'(x) = 2c_1x - 2c_2x^{-3}$

Now solve for $c_1$ and $c_2$ to find unique solution

$y(1) = c_1 \cdot 1^2 + c_2 1^{-2} = c_1 + c_2 = 3$

$y'(1) = 2c_1 \cdot 1 - 2c_2 1^{-3} = 2c_1 - 2c_2 = 2 \qquad c_1 - c_2 = 1$

$c_2 = c_1 - 1$

Eliminate

$c_1 + c_2 = 3$

$c_1 - c_2 = 1$

$2c_1 = 4, \quad c_1 = 2 \qquad c_2 = c_1 - 1 = 2 - 1 = 1$

Thus

$y(x) = 2x^2 + x^{-2}$

**2.** Use algebra and theorems on the last page to find

**(i)** $\mathcal{L}\{t^2 - e^{-9t} + 5\}(s)$

$= \mathcal{L}\{t^2\} - \mathcal{L}\{e^{-9t}\} + 5\mathcal{L}\{1\}$

by (b) with $n=2$    by (c) with $a=-9$    by (a)

$= \frac{2!}{s^3} - \frac{1}{s+9} + 5\frac{1}{s}$

$= \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$

**(ii)** $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}(t)$

$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{6}{3}\mathcal{L}^{-1}\left\{\frac{1\cdot 3}{s^2+9}\right\}$

by (e) with $k=3$    by (d) with $k=3$

$= 2\cos 3t - 2\sin 3t.$

**3.** Use the provided theorems to find

**(i)** $\mathcal{L}\{e^t \sin 3t\}(s)$.

by First Translation Theorem $a=1$

$$= \mathcal{L}\{\sin 3t\}(s-1)$$

by (d) with $k=3$

$$= \frac{3}{(s-1)^2+3^2} = \boxed{\frac{3}{(s-1)^2+9}}$$

One idea is solve the integral first and then do the transform

**(ii)** $\mathcal{L}\left\{\int_0^t e^{-\tau}\cos\tau \, d\tau\right\}(s)$.

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$f(t) = e^t \qquad g(t) = \cos t$

$$(f*g)(t) = \int_0^t e^{t-\tau}\cos\tau \, dt$$

$$= e^t \int_0^t e^{-\tau}\cos\tau \, d\tau$$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau = (g*f)(t)$$

$$\mathcal{L}\left\{\int_0^t e^{-\tau}\cos\tau\,d\tau\right\}(s) = \mathcal{L}\left\{e^{-t}(f*g)(t)\right\}(s) = \mathcal{L}\{(f*g)(t)\}(s+1)$$

first translation $a=-1$

$$= \mathcal{L}\{f(t)\}(s+1)\,\mathcal{L}\{g(t)\}(s+1) = \mathcal{L}\{e^t\}(s+1)\,\mathcal{L}\{\cos t\}(s+1)$$

(c) with $a=1$ (e) with $k=1$

$$= \left(\frac{1}{(s+1)-1}\right)\left(\frac{s+1}{(s+1)^2+1^2}\right) = \boxed{\frac{s+1}{s((s+1)^2+1)}}$$

**4.** Use the provided theorems to solve the initial-value problem

$$y'' + y = \delta(t - 2\pi) \qquad \text{where} \qquad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

$\delta$ only makes sense with Laplace transforms

$$\mathcal{L}\{y''\}(s) + \mathcal{L}\{y\}(s) = \mathcal{L}\{\delta(t-2\pi)\}(s)$$

Theorem 7.2.2 $n=2$ — $\mathcal{L}\{y\}(s) = Y(s)$ — $a = 2\pi$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = e^{-2\pi s}$$

$$s^2 Y(s) - 1 + Y(s) = e^{-2\pi s} \qquad Y(s)(s^2+1) = 1 + e^{-2\pi s}$$

$$Y(s) = \frac{1}{s^2+1} + e^{-2\pi s}\frac{1}{s^2+1}$$ ← Laplace transform of $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) + \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s^2+1}\right\}(t)$$

by (a) with $k=1$ — second translation $a = 2\pi$

$$= \sin t + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$= \sin t + \sin(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$= \sin t + \begin{cases} \sin(t-2\pi) & t \geq 2\pi \\ 0 & t < 2\pi \end{cases}$$

$$= \begin{cases} \sin t + \sin(t-2\pi) & t \geq 2\pi \\ \sin t + 0 & t < 2\pi \end{cases}$$

$$= \begin{cases} 2\sin t & \text{for } t \geq 2\pi \\ \sin t & \text{for } t < 2\pi \end{cases}$$

**5.** Write the system

$$\frac{dX}{dt} = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

without the use of matrices.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{cases} \dfrac{dx}{dy} = 4x + 2y + e^t \\[2ex] \dfrac{dy}{dt} = -x + 3y - e^t \end{cases}$$

**6.** The matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

has eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \text{and} \qquad K_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$ and $\lambda_3 = 3$. Find the general solution to $dX/dt = AX$.

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + c_3 K_3 e^{\lambda_3 t}$$

$$= c_1 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} e^{3t}$$

**THEOREM 7.1.1 Transforms of Some Basic Functions**

$\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for $a \geq 0$

(a) $\mathscr{L}\{1\} = \frac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$

(e) $\mathscr{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$

**THEOREM 7.2.1 Some Inverse Transforms**

$\mathscr{L}^{-1}\{e^{-as}\} = \delta(t-a)$ for $a \geq 0$

(a) $1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \ldots$

(c) $e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2 + k^2}\right\}$

(e) $\cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2 - k^2}\right\}$

(g) $\cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2 - k^2}\right\}$

**THEOREM 7.2.2 Transform of a Derivative**

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

**THEOREM 7.3.1 First Translation Theorem**

If $\mathscr{L}\{f(t)\} = F(s)$ and $a$ is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s - a).$$

**THEOREM 7.3.2 Second Translation Theorem**

If $F(s) = \mathscr{L}\{f(t)\}$ and $a$   0, then

$$\mathscr{L}\{f(t - a)\,\mathscr{U}(t - a)\} = e^{-as}F(s).$$

**THEOREM 7.4.1 Derivatives of Transforms**

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

**THEOREM 7.4.2 Convolution Theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

**THEOREM 7.4.3 Transform of a Periodic Function**

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period $T$, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$