Math 285: Sample Final Version A

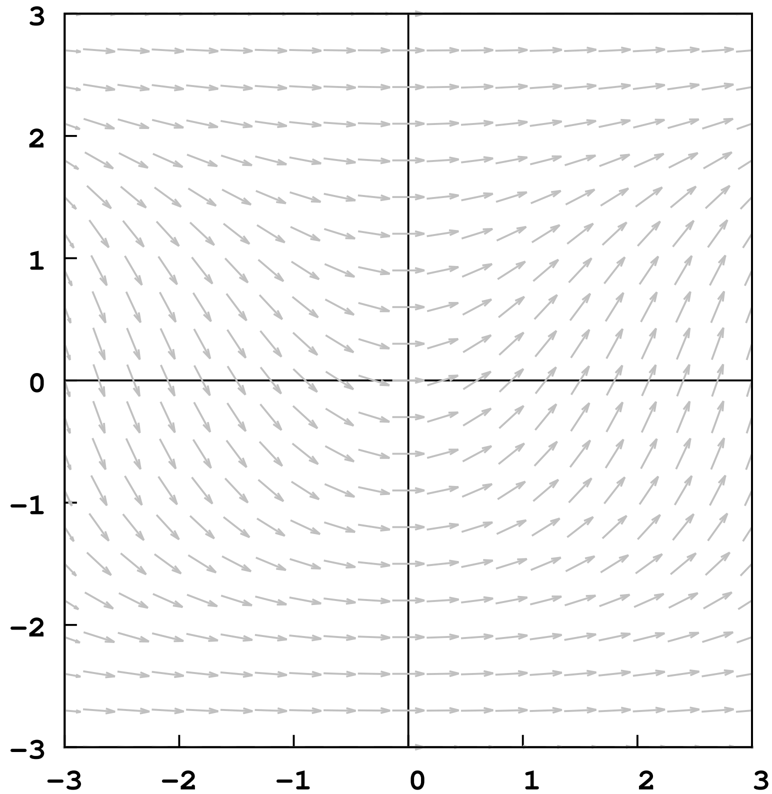
1. State the order of the given ordinary differential equations and whether the equations are linear or nonlinear.

(i) $y' = (1 - y)y$

(ii) $y'' + 11y' - 6xy = \sin x$

2. Given $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ find e^{At}.

3. The direction field for the differential equation $dy/dx = xe^{-y^2/2}$ is given. Sketch a solution curve passing through the point $y(0) = -1$.



4. Find the unique solution to the separable initial value problem

$$y' = xy^3(1 + x^2)^{1/2} \qquad \text{where} \qquad y(0) = 1.$$

5. Find the general solution to the linear differential equation

$$\frac{dy}{dx} - y = \cos 2x.$$

6. Determine the integrating factor μ depending only on y so the equation

$$y\,dx + (2x - ye^y)\,dy = 0$$

is exact. Find μ only; do not solve the differential equation.

7. Consider the differential equation

$$y' = 3x^2 + y \qquad \text{with} \qquad y(0) = 1.$$

(i) Compute one step of Euler's method with $h = 0.1$ to obtain a four-decimal approximation of $y(0.1)$.

(ii) Compute one step of the improved Euler's method with $h = 0.1$ to obtain a four-decimal approximation of $y(0.1)$.

(iii) [Extra Credit] Compute one step of the fourth-order Runge–Kutta method with $h = 0.1$ to obtain an approximation of $y(0.1)$.

8. Find the general solution to $y'' - 4y = 0$.

9. Find the general solution to $x^2 y'' + xy' + 4y = 0$.

10. Find the following Laplace and inverse Laplace transforms:

(i) $\mathcal{L}\{t^2 + 2e^{-3t} - 5\}(s)$

(ii) $\mathcal{L}^{-1}\left\{\dfrac{s+4}{s^2+9}\right\}(t)$

(iii) $\mathcal{L}\{e^{-t}\sin 3t\}(s)$.

(iv) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}(s)$.

11. Use the Laplace transform to solve

$$y'' + 16y = f(t) \qquad \text{where} \qquad f(t) = \begin{cases} \cos 4t & \text{for } 0 \le t < \pi \\ 0 & \text{for } t \ge \pi \end{cases}$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

12. The matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

has eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ 1/2 + i/2 \end{bmatrix} \qquad \text{and} \qquad K_2 = \begin{bmatrix} 1 \\ 1/2 - i/2 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = -2 + i$ and $\lambda_2 = -2 - i$.

(i) Find the general solution to $dX/dt = AX$. Use Theorem 8.2.3 to express your answer as a real soluton with real constants.

(ii) Find the unique solution to the corresponding initial value problem where $X(0) = (1, 2)$.

13. Consider the linear differential system $dX/dt = AX + F(t)$ where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \text{and} \qquad F(t) = \begin{bmatrix} -5t \\ t+5 \end{bmatrix}.$$

(i) Use the fact that A has eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad K_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$ to find a general solution X_h to the homogenous problem $dX/dt = AX$.

(ii) Use the method of undetermined coefficients to find a particular solution X_p to the original system $dX/dt = AX + F(t)$.

Summary of Numerical Methods

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n).$$

Improved Euler's Method (RK2):

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h, y_n + hk_1) \\ y_{n+1} &= y_n + (h/2)(k_1 + k_2). \end{aligned}$$

Fourth-order Runge–Kutta Method (RK4):

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1) \\ k_3 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2) \\ k_4 &= f(x_n + h, y_n + hk_3) \\ y_{n+1} &= y_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

TABLE 4.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

THEOREM 7.1.1 Transforms of Some Basic Functions

$\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for $a \geq 0$

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

THEOREM 7.2.1 Some Inverse Transforms

$\mathscr{L}^{-1}\{e^{-as}\} = \delta(t-a)$ for $a \geq 0$

(a) $1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \ldots$

(c) $e^{at} = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2+k^2}\right\}$

(e) $\cos kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2-k^2}\right\}$

(g) $\cosh kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2-k^2}\right\}$

THEOREM 7.2.2 Transform of a Derivative

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

THEOREM 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a).$$

THEOREM 7.3.2 Second Translation Theorem

If $F(s) = \mathscr{L}\{f(t)\}$ and a 0, then

$$\mathscr{L}\{f(t-a)\,\mathscr{U}(t-a)\} = e^{-as}F(s).$$

THEOREM 7.4.1 Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots,$ then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

THEOREM 7.4.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

THEOREM 7.4.3 Transform of a Periodic Function

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

THEOREM 8.2.3 Real Solutions Corresponding to a Complex Eigenvalue

Let $\lambda_1 = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $\mathbf{A}$ in the homogeneous system (2) and let $\mathbf{B}_1$ and $\mathbf{B}_2$ denote the column vectors defined in (22). Then

$$\begin{aligned} \mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t]e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t]e^{\alpha t} \end{aligned} \tag{23}$$

are linearly independent solutions of (2) on $(-\infty, \infty)$.

Note that equation (22) reads as

$$\mathbf{B}_1 = \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1) \qquad \text{and} \qquad \mathbf{B}_2 = \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1), \tag{22}$$

and equation (22) is $dX/dt = AX$.