Math 285: Sample Final Version A

**1.** State the order of the given ordinary differential equations and whether the equations are linear or nonlinear.

**(i)** $y' = (1-y)y$

$y^2$ term

order is 1

nonlinear

highest derivative is order

**(ii)** $y'' + 11y' - 6xy = \sin x$ ← ok since $x$ and not $y$

order is 2

linear

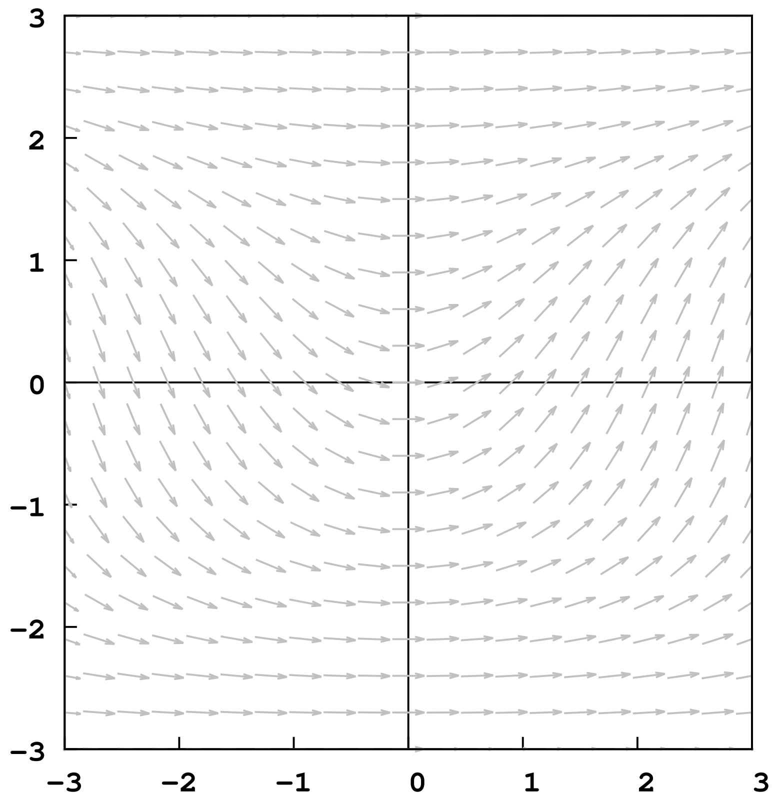
since diagonal just take exp of the diagonal terms

**2.** Given $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ find $e^{At}$. $= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}$

again diagonal

$$e^{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} t} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{0t} \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**3.** The direction field for the differential equation $dy/dx = xe^{-y^2/2}$ is given. Sketch a solution curve passing through the point $y(0) = -1$.




**4.** Find the unique solution to the separable initial value problem

$$y' = xy^3(1 + x^2)^{1/2} \qquad \text{where} \qquad y(0) = 1.$$

**5.** Find the general solution to the linear differential equation

$$\frac{dy}{dx} - y = \cos 2x.$$

**6.** Determine the integrating factor $\mu$ depending only on $y$ so the equation

$$y\,dx + (2x - ye^{y})\,dy = 0$$

is exact. Find $\mu$ only; do not solve the differential equation.

**7.** Consider the differential equation

$$y' = 3x^2 + y \qquad \text{with} \qquad y(0) = 1.$$

**(i)** Compute one step of Euler's method with $h = 0.1$ to obtain a four-decimal approximation of $y(0.1)$.

use calculator and formula on back...

**(ii)** Compute one step of the improved Euler's method with $h = 0.1$ to obtain a four-decimal approximation of $y(0.1)$.

**(iii)** [Extra Credit] Compute one step of the fourth-order Runge–Kutta method with $h = 0.1$ to obtain an approximation of $y(0.1)$.

don't spend too much time on this one because extra credit is not worth as much as a regular problem.

**8.** Find the general solution to $y'' - 4y = 0$. Guess and substitute

$$y = e^{rx} \qquad y' = re^{rx} \qquad y'' = r^2 e^{rx}$$

substitute

$$r^2 \cancel{e^{rx}} - 4\cancel{e^{rx}} = 0$$

$$r^2 - 4 = 0 \quad \text{so} \quad r = \pm 2$$

general solution is $\boxed{y(t) = c_1 e^{-2t} + c_2 e^{2t}}$

Connection between 2nd order linear equations and 1st order systems

$v = y'$ so $v' = y''$ substitute $v' - 4y = 0$

System is $\begin{aligned} y' &= v \\ v' &= 4y \end{aligned}$ $\quad X = \begin{bmatrix} y \\ v \end{bmatrix}$, $\dfrac{dX}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}}_{\text{matrix } A.} X$

**9.** Find the general solution to $x^2y'' + xy' + 4y = 0$.

homogeneous $\quad x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 0$

Guess and substitute

$$y = x^m \qquad y' = mx^{m-1} \qquad y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + x\,mx^{m-1} + 4x^m = 0$$

$$m^2 - m + m + 4 = 0 \qquad m^2 + 4 = 0 \qquad m = \pm 2i$$

General solution

$$y(t) = c_1 x^{2i} + c_2 x^{-2i}$$

recall $\quad x^{2i} = e^{2i\ln x} = \cos(2\ln x) + i\sin(2\ln x)$

$\quad x^{-2i} = e^{-2i\ln x} = \cos(2\ln x) - i\sin(2\ln x)$

real general solution

$\boxed{y(t) = \alpha_1 \cos(2\ln x) + \alpha_2 \sin(2\ln x)}$ where $\alpha_1$ and $\alpha_2$ are arbitrary constants..

**10.** Find the following Laplace and inverse Laplace transforms:

**(i)** $\mathcal{L}\{t^2 + 2e^{-3t} - 5\}(s)$

term by term

**(ii)** $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+9}\right\}(t)$

sines and cosines

**(iii)** $\mathcal{L}\{e^{-t}\sin 3t\}(s)$.

maybe translation theorem?

also translate?

**(iv)** $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}(s)$.

partial fractions?

**11.** Use the Laplace transform to solve

$$y'' + 16y = f(t) \qquad \text{where} \qquad f(t) = \begin{cases} \cos 4t & \text{for } 0 \le t < \pi \\ 0 & \text{for } t \ge \pi \end{cases}$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

easy to find $Y(s)$ then have to find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$
could be difficult

**12.** The matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

has eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ 1/2 + i/2 \end{bmatrix} \qquad \text{and} \qquad K_2 = \begin{bmatrix} 1 \\ 1/2 - i/2 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = -2 + i$ and $\lambda_2 = -2 - i$.

**(i)** Find the general solution to $dX/dt = AX$. Use Theorem 8.2.3 to express your answer as a real soluton with real constants.

**(ii)** Find the unique solution to the corresponding initial value problem where $X(0) = (1, 2)$.

**13.** Consider the linear differential system $dX/dt = AX + F(t)$ where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad F(t) = \begin{bmatrix} -5t \\ t+5 \end{bmatrix}.$$

**(i)** Use the fact that $A$ has eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$ to find a general solution $X_h$ to the homogenous problem $dX/dt = AX$.

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}$$

$$\boxed{= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}}$$

**(ii)** Use the method of undetermined coefficients to find a particular solution $X_p$ to the original system $dX/dt = AX + F(t)$.

$F(t) =$

**2.** $5x + 7$ like 5 like 7 $Ax + B$

$$X_p(t) = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} t + \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$ plug in.

$dX/dt = AX + F(t)$

$$\frac{dX_p}{dt} = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = A\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} t + \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right) + \begin{bmatrix} -5t \\ t+5 \end{bmatrix}$$ equate powers of $t$

$$0 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} t + \begin{bmatrix} -5 \\ 1 \end{bmatrix} t \qquad \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

(see back)

$$a_2 + 3b_2 = 5$$
$$3a_2 + b_2 = -1$$

$$\begin{aligned} a_2 + 3b_2 &= 5 \\ 9a_2 + 3b_2 &= -3 \\ \hline -8a_2 &= 8 \end{aligned} \qquad a_2 = -1$$

$$a_2 + 3b_2 = 5$$
$$-1 + 3b_2 = 5$$
$$3b_2 = 6 \qquad b_2 = 2$$

Thus...

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Now solve the other equation

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$a_1 + 3b_1 = 1$$
$$3a_1 + b_1 = 3$$

$$\begin{aligned} a_1 + 3b_1 &= 1 \\ 9a_1 + 3b_1 &= 9 \\ \hline -8a_1 &= -8 \end{aligned} \qquad a_1 = 1$$

$$a_1 + 3b_1 = 1$$
$$1 + 3b_1 = 1$$
$$3b_1 = 0 \qquad b_1 = 0$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution:

$$\boxed{X_p(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

## Summary of Numerical Methods

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n).$$

Improved Euler's Method (RK2):

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h, y_n + hk_1) \\ y_{n+1} &= y_n + (h/2)(k_1 + k_2). \end{aligned}$$

Fourth-order Runge–Kutta Method (RK4):

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1) \\ k_3 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2) \\ k_4 &= f(x_n + h, y_n + hk_3) \\ y_{n+1} &= y_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

**TABLE 4.4.1** Trial Particular Solutions

| $g(x)$ | Form of $y_p$ |
|---|---|
| **1.** 1 (any constant) | $A$ |
| **2.** $5x + 7$ | $Ax + B$ |
| **3.** $3x^2 - 2$ | $Ax^2 + Bx + C$ |
| **4.** $x^3 - x + 1$ | $Ax^3 + Bx^2 + Cx + E$ |
| **5.** $\sin 4x$ | $A \cos 4x + B \sin 4x$ |
| **6.** $\cos 4x$ | $A \cos 4x + B \sin 4x$ |
| **7.** $e^{5x}$ | $Ae^{5x}$ |
| **8.** $(9x - 2)e^{5x}$ | $(Ax + B)e^{5x}$ |
| **9.** $x^2e^{5x}$ | $(Ax^2 + Bx + C)e^{5x}$ |
| **10.** $e^{3x} \sin 4x$ | $Ae^{3x} \cos 4x + Be^{3x} \sin 4x$ |
| **11.** $5x^2 \sin 4x$ | $(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$ |
| **12.** $xe^{3x} \cos 4x$ | $(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$ |

**THEOREM 7.1.1 Transforms of Some Basic Functions**

$\mathscr{L}\{\delta(t-a)\} = e^{-as}$
for $a \geq 0$

**(a)** $\mathscr{L}\{1\} = \dfrac{1}{s}$

**(b)** $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

**(c)** $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

**(d)** $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$

**(e)** $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

**(f)** $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$

**(g)** $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

**THEOREM 7.2.1 Some Inverse Transforms**

$\mathscr{L}^{-1}\{e^{-as}\} = \delta(t-a)$
for $a \geq 0$

**(a)** $1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$

**(b)** $t^n = \mathscr{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \ldots$

**(c)** $e^{at} = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\}$

**(d)** $\sin kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2 + k^2}\right\}$

**(e)** $\cos kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2 + k^2}\right\}$

**(f)** $\sinh kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2 - k^2}\right\}$

**(g)** $\cosh kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2 - k^2}\right\}$

**THEOREM 7.2.2 Transform of a Derivative**

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

**THEOREM 7.3.1 First Translation Theorem**

If $\mathscr{L}\{f(t)\} = F(s)$ and $a$ is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a).$$

**THEOREM 7.3.2 Second Translation Theorem**

If $F(s) = \mathscr{L}\{f(t)\}$ and $a \quad 0$, then

$$\mathscr{L}\{f(t-a)\,\mathscr{U}(t-a)\} = e^{-as}F(s).$$

**THEOREM 7.4.1 Derivatives of Transforms**

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

**THEOREM 7.4.2 Convolution Theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

**THEOREM 7.4.3 Transform of a Periodic Function**

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period $T$, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

**THEOREM 8.2.3 Real Solutions Corresponding to a Complex Eigenvalue**

Let $\lambda_1 = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $\mathbf{A}$ in the homogeneous system (2) and let $\mathbf{B}_1$ and $\mathbf{B}_2$ denote the column vectors defined in (22). Then

$$\begin{aligned} \mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t]e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t]e^{\alpha t} \end{aligned} \tag{23}$$

are linearly independent solutions of (2) on $(-\infty, \infty)$.

Note that equation (22) reads as

$$\mathbf{B}_1 = \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1) \qquad \text{and} \qquad \mathbf{B}_2 = \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1), \tag{22}$$

and equation (22) is $dX/dt = AX$.