Math 285: Sample Midterm Version A

1. State the order of the given ordinary differential equations and whether the equations are linear or nonlinear.

(i) $\dfrac{dy}{dx} = x^2 + y^2$ nonlinear

first order

okay because in x

(ii) $y''' - 6y'' + 11y' - 6y = \sin x$

yes, linear.

third order

2. Check whether $y = e^x \sin x$ is a solution to $y'' - 2y' + 3y = 0$. Show your work explaining why or why not.

differentiate and plug in

$$y' = e^x \sin x - e^x \cos x$$

$$y'' = e^x \sin x - e^x \cos x - e^x \cos x - e^x \sin x$$

$$y'' = -2e^x \cos x$$

$$-2y' = -2e^x \sin x + 2e^x \cos x$$

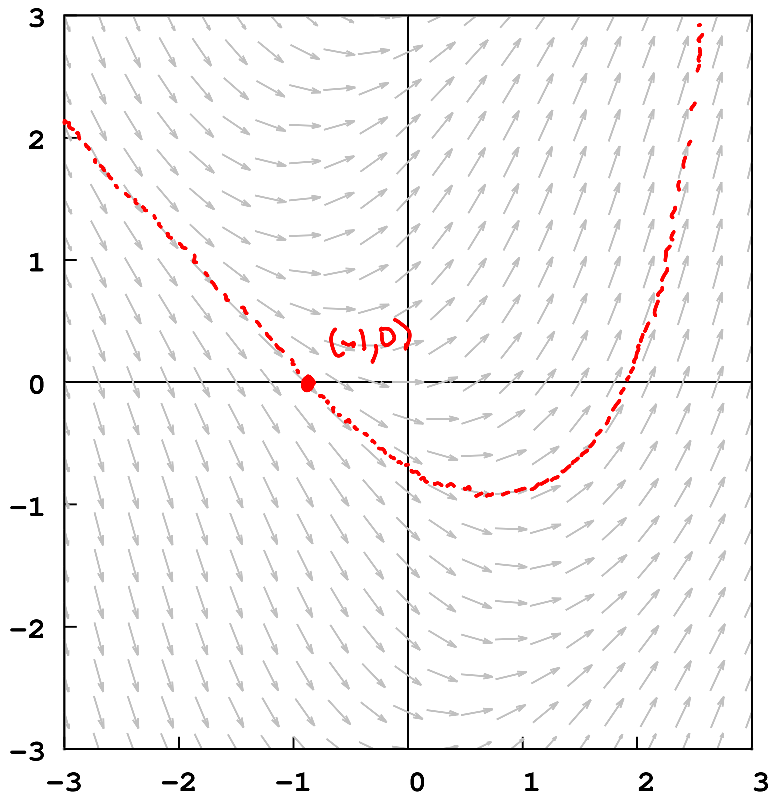
$$3y = 3e^x \sin x$$

$$y'' - 2y' + 3y = e^x \sin x \neq 0$$ so not a solution

because when you plug it in it doesn't equal,

3. The direction field for the differential equation $dy/dx = x + \sin y$ is given. Sketch a solution curve passing through the point $y(-1) = 0$.

Make sure to label where the condition $y(-1)=0$ is on the curve.



4. Use separation of variables to find an implicit solution to

$$\frac{dy}{dx} = \frac{x}{1+y^2}$$

$$\int (1+y^2)\,dy = \int x\,dx$$

$$\boxed{y + \frac{1}{3}y^3 = \frac{1}{2}x^2 + C}$$

solution written implicitly

5. Find the unique solution to the linear differential equation

$$\frac{dy}{dx} + \underbrace{y}_{P(x)y} = e^{3x} \qquad \text{such that} \qquad y(0) = 1.$$

$P(x)=1$

integrating factor: $\mu = e^{\int P(x)dx} = e^{\int 1\,dx} = e^{x}$

$$\underbrace{\frac{dy}{dx}\cdot e^{x} + y\cdot e^{x}}_{\text{combine by product rule}} = e^{3x}\cdot e^{x}$$

$\frac{d}{dx}(ye^{x}) = e^{4x}$ so $ye^{x} = \int e^{4x}dx = \frac{1}{4}e^{4x} + C$

divide by e^{x}

so $y = \frac{1}{4}e^{3x} + Ce^{-x}$ is general solution... Solve for C to find unique solution

$y(0) = \frac{1}{4}e^{3\cdot 0} + Ce^{-0} = \frac{1}{4} + C = 1$ so $C = 1 - \frac{1}{4} = \frac{3}{4}$.

unique solution $y = \frac{1}{4}e^{3x} + \frac{3}{4}e^{-x}$

6. Determine whether the functions

$$f_1(x) = 1 + x, \qquad f_2(x) = x \qquad \text{and} \qquad f_3(x) = x^2$$

are linearly independent on the interval $(-\infty, \infty)$.

Linear independence means that if $c_1f_1 + c_2f_2 + c_3f_3 = 0$ then all the c_i's must be 0.

$c_1(1+x) + c_2(x) + c_3x^2 = 0$ implies $c_1 = c_2 = c_3 = 0$ but why.

$$W = \det\begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix} = \det\begin{bmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$

$$= -\det\begin{bmatrix} 1 & 1 & 2x \\ 1+x & x & x^2 \\ 0 & 0 & 2 \end{bmatrix} = -\det\begin{bmatrix} 1 & 1 & 2x \\ 0 & -1 & x^2-(1+x)2x \\ 0 & 0 & 2 \end{bmatrix} = 2$$

$r_2 \leftarrow r_2 - (1+x)r_1$

not zero so linearly indep.

Alternative solution:

6. Determine whether the functions

$$f_1(x) = 1 + x, \qquad f_2(x) = x \qquad \text{and} \qquad f_3(x) = x^2$$

are linearly independent on the interval $(-\infty, \infty)$.

Linear independence means that if

$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$ then all the c_i's must be 0.

$\underbrace{c_1(1+x) + c_2(x) + c_3 x^2}$ $= 0$ implies $c_1 = c_2 = c_3 = 0$

but why?

if a polynomial is zero, the all coef of powers of x are zero

$$c_1 + (c_1 + c_2)x + c_3 x^2 = 0$$

This means $c_1 = 0$ $\quad c_1 + c_2 = 0$ $\quad$ and $c_3 = 0$

$0 + c_2 = 0$

so $c_2 = 0$ $\quad$ then $c_1 = c_2 = c_3 = 0 \ldots$

7. Substitute $y = xu$ to reduce the homogeneous equation

$$(y^2 + yx)dx + x^2 dy = 0$$

to a separable differential equation in u. Don't solve the equation in u.

$y = xu \qquad dy = (dx)u + x\,du$

$$((xu)^2 + xu \cdot x)dx + x^2(u\,dx + x\,du) = 0$$

$$x^2(u^2+u)dx + x^2 u\,dx + x^3 du = 0$$

$$x^2(u^2+2u)dx = -x^3 du$$

$$\boxed{\frac{dx}{x} = \frac{-1}{u^2+2u}du}$$

← no going to do the integrals

8. Solve the differential equation

$$(2y^2 + 3x)\,dx + 2xy\,dy = 0$$

by finding an integrating factor μ that depends only on x so it is exact.

$$\mu(x)(2y^2+3x)dx + \mu(x)2xy\,dy = 0$$

$$\frac{\partial}{\partial y}\left(\mu(x)(2y^2+3x)\right) = \frac{\partial}{\partial x}\left(\mu(x)2xy\right), \qquad \mu = \mu(x)$$

$$\mu \cdot (4y) = \mu' \cdot 2xy + \mu \cdot 2y$$

$$4\mu = 2x\mu' + 2\mu$$

$$2x\mu' - 2\mu = 0 \qquad x\mu' - \mu = 0 \qquad x\frac{d\mu}{dx} = \mu$$

$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x} \qquad \ln\mu = \ln x + C \qquad \mu = e^{\ln x + C} = Ax$$

$\mu(x)(2y^2+3x)dx + \mu(x)2xy\,dy = 0$

$(2y^2x + 3x^2)dx + 2x^2y\,dy = 0$

$\frac{\partial f}{\partial x} = (2y^2x + 3x^2)$ $\quad f(x,y) = y^2x^2 + x^3 + g(y)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^2x^2 + x^3 + g(y)) = \cancel{2yx^2} + 0 + g'(y) = \cancel{2x^2y}$

$g'(y) = 0$ so $g(y) = C$

answer $\boxed{y^2x^2 + x^3 = C}$

9. Consider the differential equation

$$y' = 4x - 2y \qquad \text{with} \qquad y(0) = 2.$$

(i) Compute one step of Euler's method with $h = 0.1$ to obtain a four-decimal approximation of $y(0.1)$.

$$y(0.1) \approx y_1 = y_0 + hf(x_0, y_0) = 2 + (0.1)(4\cdot 0 - 2\cdot 2) = 2 - .4 = \boxed{1.6000}$$

(ii) Compute one step of the improved Euler's method with $h = 0.1$ to obtain a four-decimal approximation of $y(0.1)$.

$$k_1 = f(x_0, y_0) = 4\cdot 0 - 2\cdot 2 = -4$$

$$k_1 = f(x_0 + h, y_0 + hk_1) = 4\cdot 0.1 - 2(1.6) = .4 - 3.2 = -2.8$$

$$y(0.1) \approx y_1 = y_0 + \frac{h}{2}(k_1 + k_2) = 2 + (0.05)(-4 - 2.8) = \boxed{1.6600}$$

(iii) Compute one step of the fourth-order Runge–Kutta method with $h = 0.1$ to obtain a four-decimal approximation of $y(0.1)$.

$$K_1 = f(x_0, y_0) = 4\cdot 0 - 2\cdot 2 = -4$$

$$K_2 = f(x_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2}K_1) = 4\cdot 0.05 - 2\cdot(2 - 0.05\cdot 4) = -3.4$$

$$K_3 = f(x_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2}K_2) = 4\cdot 0.05 - 2(2 - 0.05\cdot 3.4) = -3.46$$

$$K_4 = f(x_0 + h, y_0 + hK_3) = 4\cdot 0.1 - 2(2 - 0.1\cdot 3.46) = -2.908$$

$$y(0.1) \approx y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2 - \frac{0.1}{6}(4 + 2\cdot 3.4 + 2\cdot 3.46 + 2.908)$$

$$\boxed{= 1.6562}$$

10. The function $y_1(x) = x^4$ satisfies $x^2y'' - 7xy' + 16y = 0$ for $x > 0$. Use reduction of order to find a second solution $y_2(x)$.

$y = x^4 u \qquad y' = x^4 u' + 4x^3 u \qquad y'' = x^4 u'' + \underbrace{4x^3 u' + 4x^3 u'}_{8x^3u'} + 12x^2 u$

$$x^2 y'' = x^6 u'' + 8x^5 u' + 12x^4 u$$
$$-7xy' = -7x^5 u' - 28x^4 u$$
$$16y = 16x^4$$
$$0 = x^6 u'' + x^5 u'$$

$v = u'$ so $\underbrace{xv' + v}_{\text{combine}} = 0$

$\frac{d}{dx}(xv) = 0$ so $xv = c_1$

so $v = \frac{c_1}{x}$

$u' = \frac{c_1}{x}$

$u = \int \frac{c_1}{x}\,dx = c_1 \ln x + c_2$

$y = x^4 u = x^4(c_1 \ln x + c_2)$

doesn't give a new soln

Second solution is

$$\boxed{y_2 = x^4 \ln x}$$

11. Solve $y'' - 2y' + 5y = e^x$ by undetermined coefficients.

Solve homogeneous solution: $r^2 - 2r + 5 = 0$

plug in $y = e^{rx}$ $\qquad a = 1 \quad b = -2 \quad c = 5$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$y_h = c_1 e^x \cos 2x + c_2 e^x \sin 2x$

Solve particular solution

$y_p = Ae^x \qquad y_p' = Ae^x \qquad y_p'' = Ae^x$

$$Ae^x - 2Ae^x + 5Ae^x = e^x$$

$$A - 2A + 5A = 1$$

$4A = 1 \qquad A = \frac{1}{4}$

Solution $\boxed{y = c_1 e^x \cos 2x + c_2 e^x \sin 2x + \frac{1}{4} e^x}$

Summary of Numerical Methods

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n).$$

Improved Euler's Method (RK2):

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h, y_n + hk_1) \\ y_{n+1} &= y_n + (h/2)(k_1 + k_2). \end{aligned}$$

Fourth-order Runge–Kutta Method (RK4):

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1) \\ k_3 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2) \\ k_4 &= f(x_n + h, y_n + hk_3) \\ y_{n+1} &= y_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

TABLE 4.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$