§1.3#1. For each of the following functions f, show that

$$\lim_{(x,y)\to(0,0)} f(x,y) \qquad (*)$$

does not exist.

(a) $f(x,y) = \dfrac{x^2+y}{\sqrt{x^2+y^2}}$

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{\sqrt{x^2}} = \lim_{x\to 0} |x| = 0$$

$$\lim_{y\to 0^+} f(0,y) = \lim_{y\to 0^+} \frac{y}{\sqrt{y^2}} = \lim_{y\to 0^+} \frac{y}{|y|} = 1$$

Since the limits taken along different axis are different then the limit $(*)$ does not exist.

(b) $f(x,y) = \dfrac{x}{x^4+y^4}$

If it existed, then it would exist taken along the x-axis. However,

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x}{x^4} = \lim_{x\to 0} \frac{1}{x^3} = DNE$$

so the limit $(*)$ does not exist.

§1.3 #1 continues...

(e) $f(x,y) = \dfrac{x^4y^4}{(x^2+y^4)^3}$

If the limit exists, then it would exist along the curve $x = cy^2$ and not depend on the value of c, but

$$\lim_{y\to 0} f(cy^2, y) = \lim_{y\to 0} \frac{c^4 y^{12}}{(c^2y^4+y^4)^3} = \frac{c^4}{c^2+1}$$

does depend on c, so the limit (1) does not exist.

§1.3 #2 For the following functions f, show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0. \qquad (**)$$

(a) $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$.

To show this we use the following inequality

$$|xy| \le \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2,$$

which can be derived as.

$$(x-y)^2 \ge 0 \qquad\qquad (x+y)^2 \ge 0$$
$$x^2 - 2xy + y^2 \ge 0 \qquad\qquad x^2 + 2xy + y^2 \ge 0$$
$$xy \le \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 \qquad\qquad -xy \le \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2$$

Therefore $|xy| \le \frac{1}{2}x^2 + \frac{1}{2}y^2$.

Now

$$|f(x,y)| = \left|\frac{x^2y^2}{x^2+y^2}\right| \le \frac{\left(\frac{1}{2}x^2+\frac{1}{2}y^2\right)^2}{x^2+y^2} = \frac{1}{4}\|(x,y)\|_2^2$$

Let $\varepsilon > 0$. Then choose $\delta = \frac{1}{2}\sqrt{\varepsilon}$. It follows that for $\|(x,y)-(0,0)\| < \delta$ we have

$$|f(x,y)| < \tfrac{1}{4}\delta^2 = \varepsilon.$$

Since ε is arbitrary $(**)$ holds.

§1.3#2 continues...

(b) $f(x,y) = \dfrac{3x^5 - xy^4}{x^4+y^4}$

Now

$$|f(x,y)| = |x|\left|\frac{3x^4-y^4}{x^4+y^4}\right| \le |x|\,\frac{3x^4+3y^4}{x^4+y^4} = 3|x|$$

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. Then

$$|x| \le \sqrt{x^2+y^2} = \|(x,y)-(0,0)\|_2 < \delta$$

implies $|f(x,y)| < 3\delta = \varepsilon$. Since ε is arbitrary then (**) holds.

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§2.2#1. For each of the following functions f, (i) compute ∇f, (ii) find the directional derivative of f at the point $(1,-2)$ in the direction $(3/5, 4/5)$.

(a) $f(x,y) = x^2y + \sin \pi xy$.

(i) $\nabla f = (2xy + \pi y \cos \pi xy,\ x^2 + \pi x \cos \pi xy)$

(ii) $\nabla f(1,-2) = (-4 - 2\pi \cos(-2\pi),\ 1 + \pi \cos(-2\pi))$

$= (-4 - 2\pi,\ 1+\pi)$

$\nabla f(1,-2) \cdot (\frac{3}{5}, \frac{4}{5}) = \frac{(-4-2\pi)3}{5} + \frac{(1+\pi)4}{5}$

$= \frac{-8-2\pi}{5}$

(b) $f(x,y) = e^{4x-y^2}$

(i) $\nabla f = (4e^{4x-y^2},\ -2y\, e^{4x-y^2})$

(ii) $\nabla f(1,-2) = (4e^0,\ -2(-2)e^0) = (4,4)$

$\nabla f(1,-2) \cdot (\frac{3}{5}, \frac{4}{5}) = \frac{12}{5} + \frac{16}{5} = \frac{28}{5}$

(c) $f(x,y) = \frac{x+2y+4}{7x+3y}$

(i) $\nabla f = \left(\frac{(7x+3y) - 7(x+2y+4)}{(7x+3y)^2},\ \frac{2(7x+3y) - 3(x+2y+4)}{(7x+3y)^2} \right)$

$= \left(\frac{-11y-28}{(7x+3y)^2},\ \frac{11x-12}{(7x+3y)^2} \right)$

§2.2#1 continues...

(c) continues...

$$\nabla f(1,-2) = \left(\frac{22-28}{1^2}, \frac{-1}{1^2}\right) = (-6,-1)$$

$$\nabla f(1,-2)\cdot\left(\frac{3}{5},\frac{4}{5}\right) = (-6,-1)\cdot\left(\frac{3}{5},\frac{4}{5}\right) = \frac{-18-4}{5} = \frac{-22}{5}$$

§3.2#1 For each of the following functions $F(x,y)$, determine wether the set $S=\{(x,y): F(x,y)=0\}$ is a smooth curve. Draw a sketch of S. Examine the nature of S near any points where $\nabla F=0$. Near which points of S is S the graph of a function $y=f(x)$? $x=f(y)$?

(a) $F(x,y)=x^2+3y^2-5$.

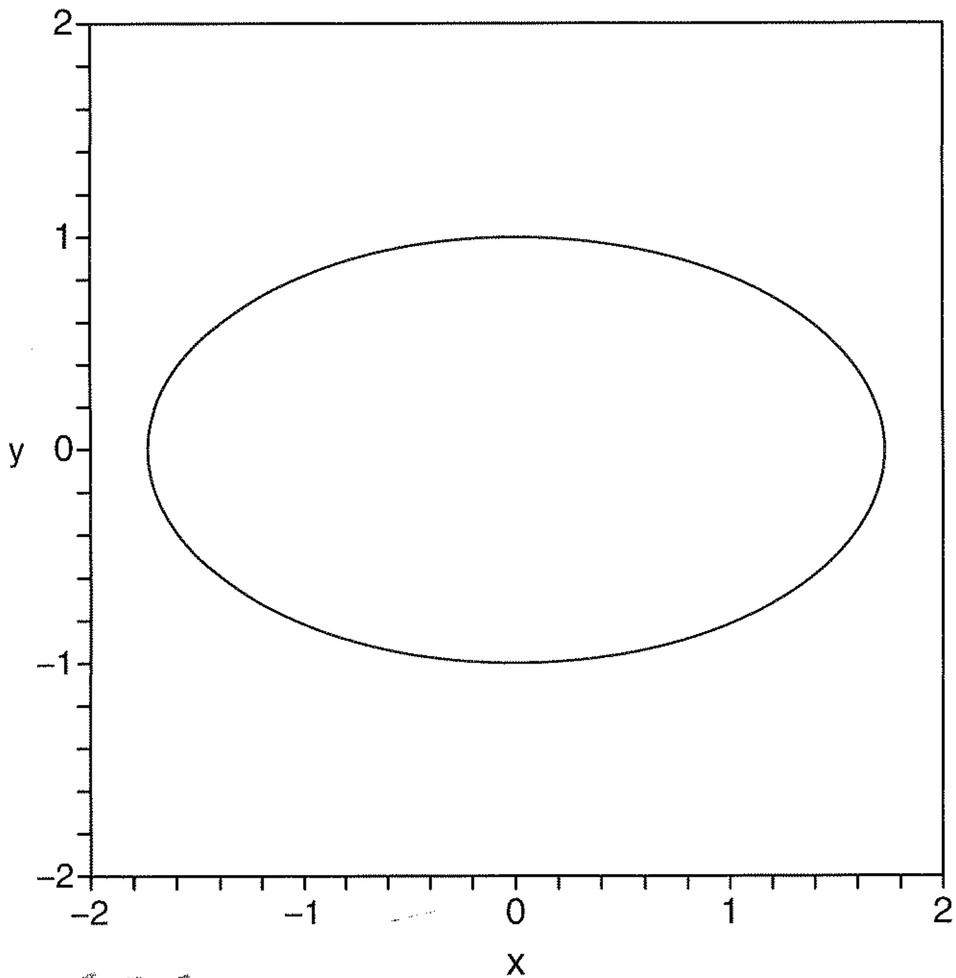
S is an ellipse

```
> restart;
```

```
> with(plots):
  doplot:=(f,a,b,c,d)->contourplot(f,x=a..b,y=c..d,contours=[0],
        numpoints=10000,axes=boxed,view=[a..b,c..d]);
```

$$doplot := (f, a, b, c, d) \rightarrow contourplot(f, x = a .. b, y = c .. d, contours = [0], numpoints = 10000, axes = boxed, view = [a .. b, c .. d])$$

```
> #3.2.1a
  doplot(x^2+3*y^2-3,-2,2,-2,2);
```



(a) $F(x,y) = x^2 + 3y^2 - 3$

(i) S is an ellipse.

(ii) $\nabla F = (2x, 6y) = 0$ when $x=0$, $y=0$. This point is not on S so $\nabla F \neq 0$ for all $(x,y) \in S$.

(iii) S is locally a graph of the form $y = f(x)$ provided that $\partial F/\partial y \neq 0$. This is points $(x,y) \in S$ s.t. $y \neq 0$

(iv) S is locally a graph $x = f(y)$ for points $(x,y) \in S$ st $x \neq 0$.

(v) S is a smooth curve.

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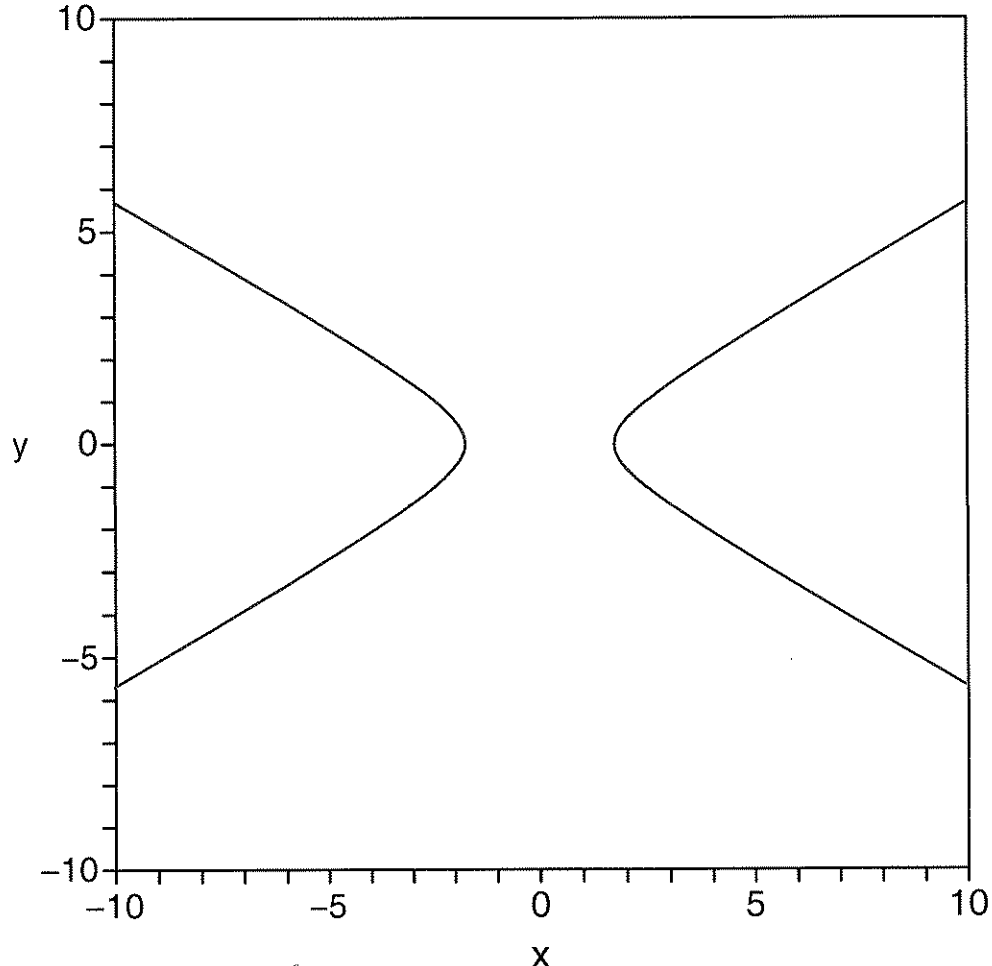
```
> #3.2.1b
doplot(x^2-3*y^2-3,-10,10,-10,10);
```



(b) $F(x,y) = x^2 - 3y^2 - 3$

(i) S is a hyperbola.

(ii) $\nabla F = (2x, -6y) = 0$ when $(x,y) = (0,0)$. This point is not on S. So $\nabla F \neq 0$ for all $(x,y) \in S$.

(iii) S is locally $y = f(x)$ provided $y \neq 0$

(iv) S is locally $x = f(y)$ provided $x \neq 0$.

(v) S is not a smooth curve because it is not connected.

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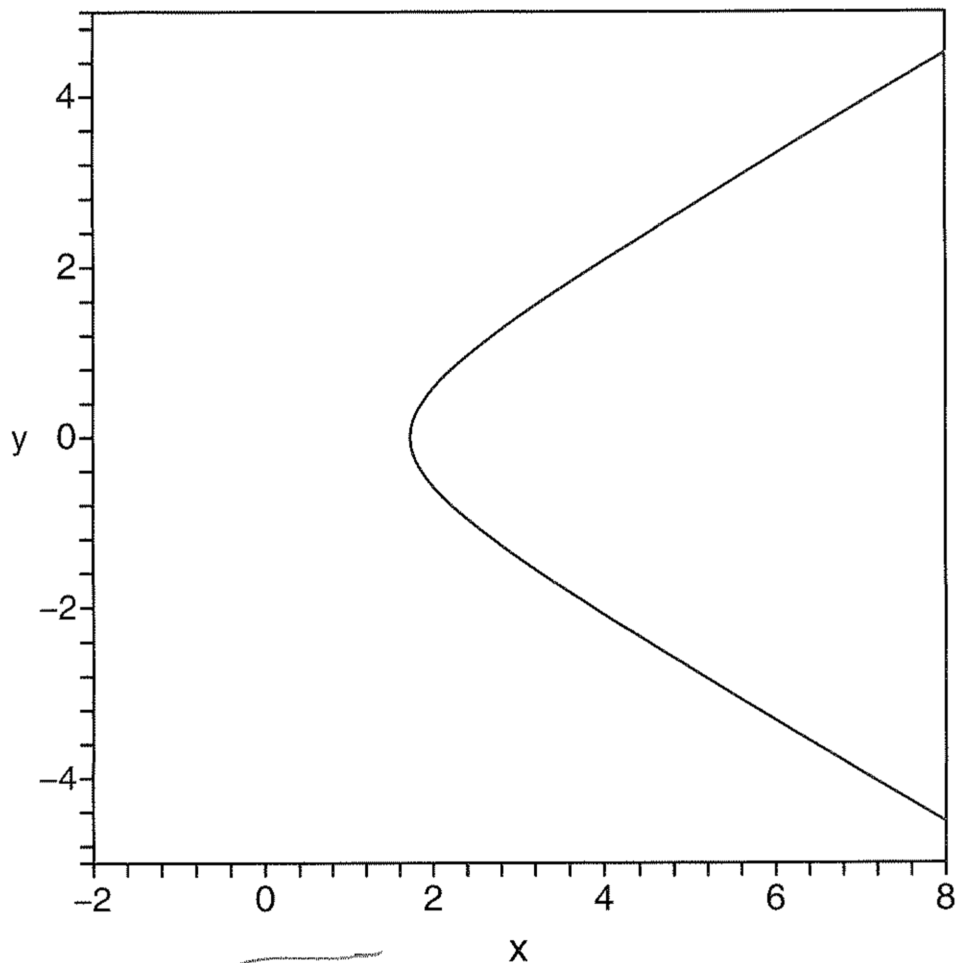
```
> #3.2.1c
  doplot(x-sqrt(3*(y^2+1)),-2,8,-5,5);
```



(c) $F(x,y) = x - \sqrt{3(y^2+1)}$

(i) S is half of a hyperbola.

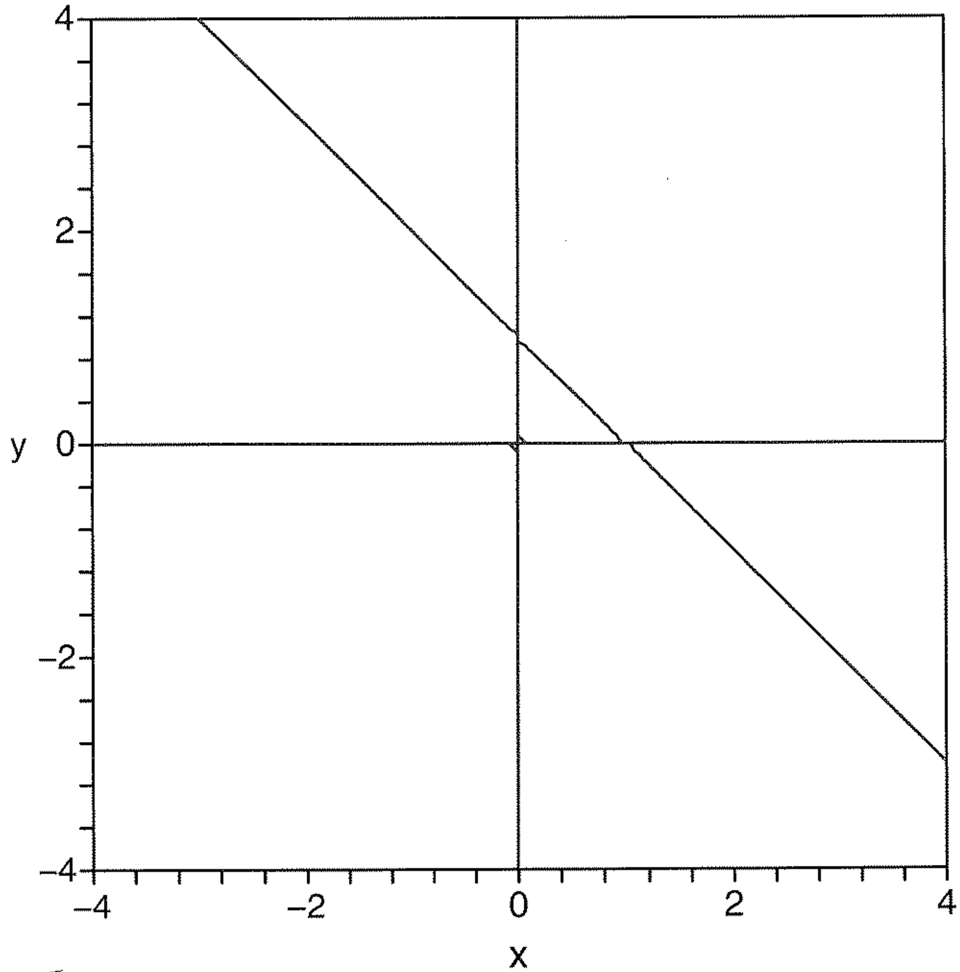
(ii) $\nabla F = \left(1, -\frac{1}{2}(3y^2+1)^{-1/2} 6y\right) \neq 0$ for all $(x,y) \in S$.

(iii) S is locally $y = f(x)$ near (x,y) provided $y \neq 0$.

(iv) S is locally $x = f(y)$ for all points.

(v) S is a smooth curve.

```
> #3.2.1d
  doplot(x*y*(x+y-1),-4,4,-4,4);
```



(d) $F(x,y) = xy(x+y-1)$

(i) S is the union of 3 straight lines

(ii) $\nabla F = (y(x+y-1)+xy,\ x(x+y-1)+xy) = 0$

when $\begin{cases} y(2x+y-1)=0 \\ x(x+2y-1)=0 \end{cases}$

If $y=0$ then $x(x-1)=0$ and $\nabla F=0$ at $(0,0)$ and $(1,0)$

If $y \neq 0$ then $2x+y-1=0$, $y=1-2x$ and therefore $x(x+2(1-2x)-1)=0$ or $x(1-3x)=0$

thus $x=0$ or $\frac{1}{3}$. Hence $\nabla F=0$ at $(0,1)$ and $(\frac{1}{3},\frac{1}{3})$

However $(\frac{1}{3},\frac{1}{3}) \notin S$. The points where $\nabla F=0$ on S are $(0,0)$, $(1,0)$ and $(0,1)$. These are the places where the lines cross.

§3.2 #1d continues...

(iii) S is locally $y=f(x)$ near (x,y) provided $\frac{\partial f}{\partial y} \neq 0$.

$\frac{\partial f}{\partial y} = x(x+2y-1) = 0$

when either $x=0$ or $x=1-2y$.

The only point $(1-2y, y) \in S$ is when

$(1-2y)y(1-2y+y-1) = 0$

so $y=0$, $y=\frac{1}{2}$

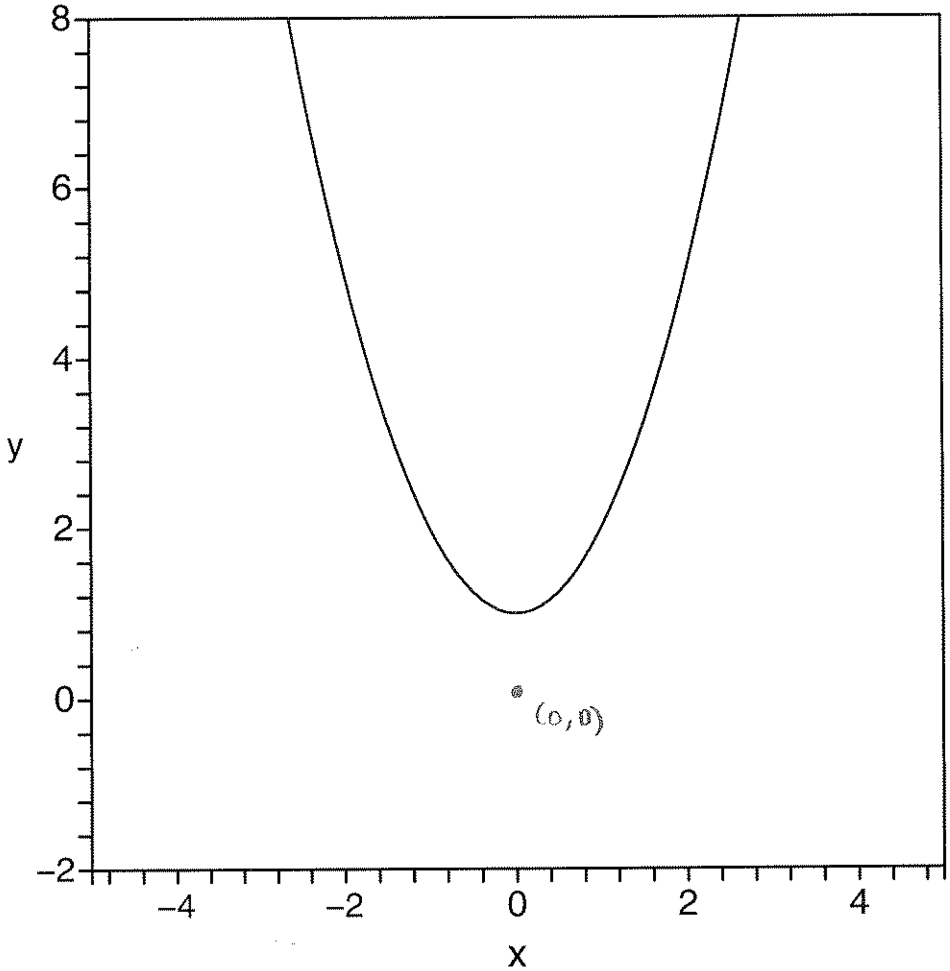
Thus $(0,y)$ or $(1,0)$ are the points in S where $\partial f/\partial y = 0$.

S is locally $y=f(x)$ for points $(x,y) \in S$ where both $x \neq 0$ and $(x,y) \neq (1,0)$.

(iv) S is locally $x=f(y)$ for points $(x,y) \in S$ where both $y \neq 0$ and $(x,y) \neq (0,1)$ by symmetry.

(v) S is not a smooth curve, because at the points $(0,0)$, $(0,1)$ and $(1,0)$ it is neither a graph of the form $y=f(x)$ or $x=f(y)$.

```
> #3.2.1e
doplot((x^2+y^2)*(y-x^2-1),-5,5,-2,8);
```



(e) $F(x,y) = (x^2+y^2)(y-x^2-1)$

(i) This the a parabola plus the point at $(0,0)$.

(ii) $\nabla F = \left(2x(y-x^2-1)-2x(x^2+y^2),\ 2y(y-x^2-1)+(x^2+y^2)\right) = 0$

$$\begin{cases} 2x(y-y^2-2x^2-1) = 0 \\ 3y^2-2yx^2-2y+x^2 = 0 \end{cases}$$

If $x=0$ then $y(3y-2)=0$ so $y=0$ or $y=\frac{2}{3}$.

Since $(0,\frac{2}{3}) \notin S$ then $\nabla F = 0$ for $(0,0)$

If $x \neq 0$ then $2x^2 = y-y^2-1$ and so to be is S

$y - \frac{y-y^2-1}{2} - 1 = 0$ or $2y - y + y^2 + 1 - 2 = 0$

Thus $y^2+y-1=0$ so $y = \frac{-1\pm\sqrt{5}}{2}$.

§3.2#1e. continues ...

Now if $y = \frac{-1+\sqrt{5}}{2}$ then $2x^2 = y - y^2 - 1$ implies that $2x^2 = -3+\sqrt{5} < 0$

if $y = \frac{-1-\sqrt{5}}{2}$ then $2x^2 = -3-\sqrt{5} < 0$

which are both impossibilities thus the only point on S where $\nabla F = 0$ is $(0,0)$.

(iii) S is locally $y = f(x)$ at points $(x,y) \in S$ such that $(x,y) \neq 0$

(iv) S is locally $x = f(y)$ at points $(x,y) \in S$ such that $(x,y) \neq 0$ and $(x,y) \neq (0,1)$.

(v) S is not a smooth curve because it is not connected. Also the point at $(0,0)$ is not the graph of either $y = f(x)$ or $x = f(y)$.

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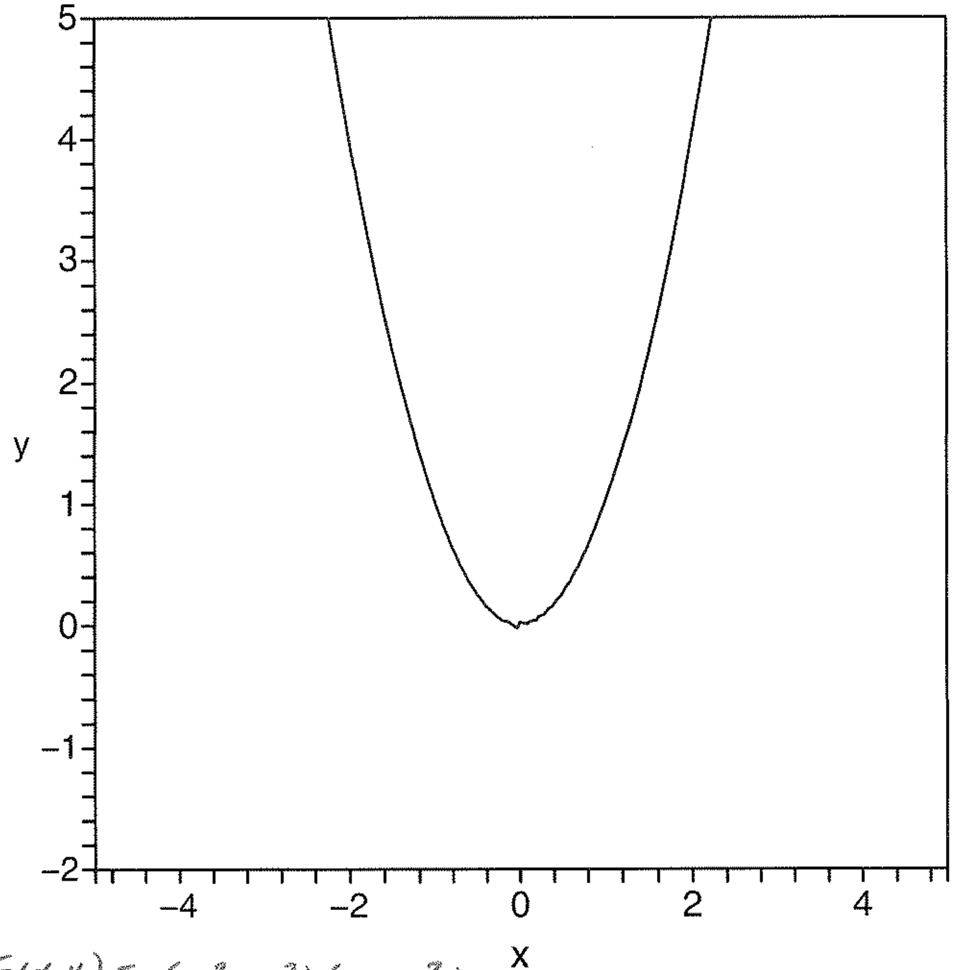
```
> #3.2.1f
  doplot((x^2+y^2)*(y-x^2),-5,5,-2,5);
```



(f) $F(x,y) = (x^2+y^2)(y-x^2)$

(i) S is a parabola.

(ii) $\nabla F = (2x(y-x^2) - 2x(x^2+y^2),\ 2y(y-x^2) + x^2+y^2) = 0$

when $x=0$ then $2y^2+y^2=0$ so $y=0$.

If $x \neq 0$ then $y-3x^2-2y^2=0$ so $3x^2=2y^2-y$.

For such a point to lie on S we have.

$3y-2y^2+y=0$ or $y(1-y)=0$ so $y=0$ or $y=1$

If $y=0$ then $x=0$ a contradiction thus $y=1$

In this case $x^2=1/3$. But then $\partial F/\partial y = 2(1-\frac{1}{3})+1+\frac{1}{3} = 2\frac{2}{3}$

and so $\nabla F \neq 0$.

Thus the only point $(x,y) \in S$ when $\nabla F = 0$ is $(0,0)$.

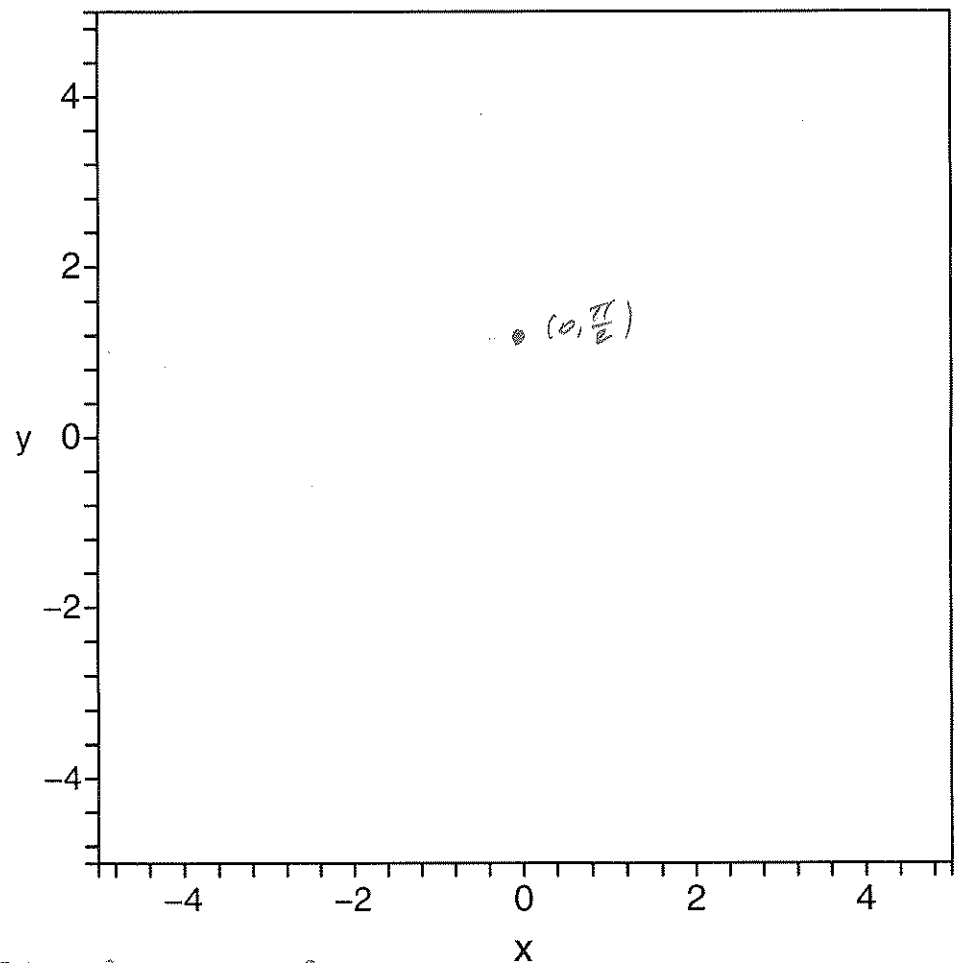
§3.2 #18 continues...

(iii) S is locally a graph of the form $y=f(x)$ for any point $(x,y)\in S$.

(iv) S is locally a graph of the form $x=f(y)$ for any point $(x,y)\in S$ except $(0,0)$.

(v) S is a smooth curve.

```
> #3.2.1g
  doplot((exp(x)-1)^2+(sin(y)-1)^2,-5,5,-5,5);
```



Pf) $F(x,y) = (e^x-1)^2 + (\sin y - 1)^2$

(i) S is a collection of points of the form $(0, \frac{\pi}{2}+2k\pi)$ for $k \in \mathbb{Z}$.

(ii) $\nabla F = (2(e^x-1)e^x,\ 2(\sin y-1)\cos y) = 0$ when $x=0$ and $y = \frac{\pi}{2}+2k\pi$.

(iii)–(iv) S is not locally $y=f(x)$ or $x=f(y)$ at any points

(v) S is not a smooth curve.

§3.2#5 Let $f(t) = \left(\frac{t^2-1}{t^2+1}, \frac{t(t^2-1)}{t^2+1}\right)$ and $S = \{f(t) : t \in \mathbb{R}\}$.

(a) Show that S is the locus of the equation

$$y^2(1-x) = x^2(1+x)$$

Let $L = \{(x,y) : y^2(1-x) = x^2(1+x)\}$

Claim $L = S$.

"$\supseteq$"

$$y^2(1-x) = \left(t\,\frac{t^2-1}{t^2+1}\right)^2\left(1 - \frac{t^2-1}{t^2+1}\right)$$

$$= t^2\left(\frac{t^2-1}{t^2+1}\right)^2 \frac{t^2+1-t^2+1}{t^2+1}$$

$$= t^2\left(\frac{t^2-1}{t^2+1}\right)\frac{2}{t^2+1} = 2t^2\frac{(t^2-1)^2}{(t^2+1)^3}$$

$$x^2(1-x) = \left(\frac{t^2-1}{t^2+1}\right)^2\left(1 + \frac{t^2-1}{t^2+1}\right)$$

$$= \frac{(t^2-1)^2}{(t^2+1)^2}\;\frac{t^2+\cancel{1}+t^2-\cancel{1}}{t^2+1}$$

$$= 2t^2\,\frac{(t^2-1)^2}{(t^2+1)^3}$$

Therefore every point $(x,y) \in S$ satisfies the equation $y^2(1-x) = x^2(1+x)$ and so $S \subseteq L$.

"$\subseteq$" Suppose $y^2(1-x) = x^2(1+x)$.

Case $x=1$. Then $x^2(1+x)=0$ and so $x=0$ or $x=-1$, both are contradictions so $x \neq 1$.

Case $x=-1$ then $y^2(1-x)=0$ and so $y=0$. In this case $t=0$ gives

$$f(0) = \left(\frac{-1}{1}, 0(-1)(1)\right) = (-1,0) = (x,y)$$

and so $(x,y) \in S$.

Case $x \in \mathbb{R} \setminus \{-1,1\}$. Then $y^2 = x^2 \frac{1+x}{1-x}$ and therefore

$$\frac{1+x}{1-x} \geq 0.$$

Since $x \neq -1$, then $\frac{1+x}{1-x} > 0$ and it follows that $x \in (-1,1)$.

Define $g: (0,\infty) \to \mathbb{R}$ by $g(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1}$.

Since $x \in (-1,1)$ then $\frac{1+x}{1-x} > 0$. It follows that

$$g\left(\frac{1-x}{x+1}\right) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{1+x-1+x}{1+x+1-x} = x$$

Thus, $(-1,1) \subseteq g((0,\infty))$.

§3.2#5 continues...

Claim that $g:(0,\infty)\to\mathbb{R}$ is 1-to-1.

Suppose $s_1, s_2 \in (0,\infty)$ such that

$g(s_1) = g(s_2)$. Then

$$1-\frac{2}{s_1+1} = 1-\frac{2}{s_2+1}$$

implies $s_1 = s_2$. Thus g is 1-to-1.

Therefore

$$s = \frac{1+x}{1-x}$$

is the unique $s\in(0,\infty)$ such that $g(s)=x$.

Subcase: $xy>0$. Then take $t=\sqrt{s}$.
It follows that $t^2=s$ and thus

$$x = g(s) = g(t^2) = \frac{t^2-1}{t^2+1}$$

Now

$$t\,\frac{t^2-1}{t^2+1} = \sqrt{s}\,\frac{s-1}{s+1} = \sqrt{\frac{1+x}{1-x}}\;\frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}$$

$$= x\sqrt{\frac{1+x}{1-x}} = x\sqrt{\frac{y^2}{x^2}} = x\,\frac{|y|}{|x|}$$

$$= \begin{cases} |y| & \text{if } x>0 \\ -|y| & \text{if } x<0 \end{cases}$$

$$= \begin{cases} y & \text{if } x>0,\ y>0 \\ y & \text{if } x<0\ \ y<0 \end{cases} = y$$

Thus $(x,y)=f(t)$ so $(x,y)\in S$.

Subcase $xy<0$. Then take $t=-\sqrt{5}$.
It follows that $t^2=5$ and thus

$$x=\frac{t^2-1}{t^2+1}.$$

Now

$$t\,\frac{t^2-1}{t^2+1}=-\sqrt{5}\,\frac{5-1}{5+1}=\begin{cases}-|y| & \text{if } x>0\\ |y| & \text{if } x<0\end{cases}$$

$$=\begin{cases}y & \text{if } x>0 \text{ and } y<0\\ y & \text{if } x<0 \text{ and } y>0\end{cases}$$

$$=y$$

Thus $(x,y)=f(t)$ and so $(x,y)\in S$.

In all cases $(x,y)\in L$ implies $(x,y)\in S$ and therefore $L\subseteq S$.

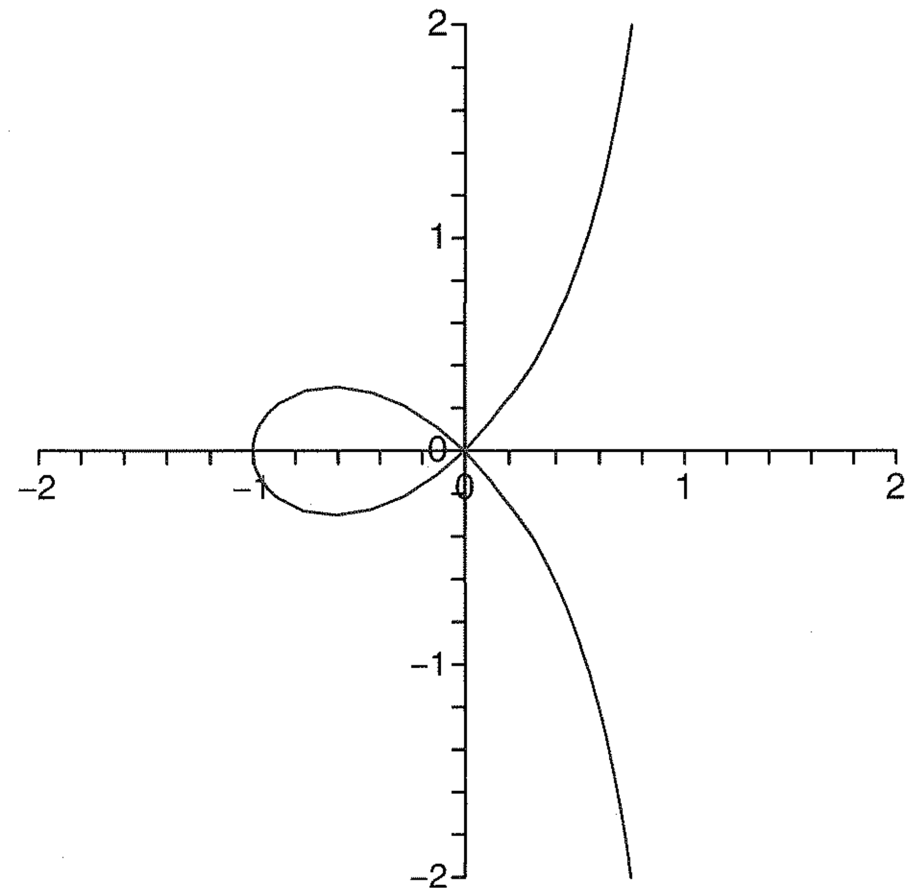
It follows that $L=S$.

§3.2#56. Draw a sketch of S. Show that S is asymptotic to the line x=1

```
> restart;
> f:=t->((t^2-1)/(t^2+1),t*(t^2-1)/(t^2+1));
```

$$f := t \to \frac{t^2 - 1}{t^2 + 1}, \frac{t\,(t^2 - 1)}{t^2 + 1}$$

```
> plot([f(t),t=-3..3],-2..2,-2..2);
```



§3.2#5b continues.

To show it is asymptotic we need show that $x \to 1$ as $y \to \infty$ or $y \to -\infty$.

We have already shown that $x=1$ is not a point on the curve. Now since $x \in [-1,1)$ and

$$y^2 = x^2 \frac{1+x}{1-x} \to \infty \quad \text{as } y \to \infty$$

Then it follows that $x \to 1$.

§3.2#5c. Discuss the nature of the point $(0,0)$ where S crosses itself in terms of the parametric and the non parametric representations of S in (a).

Non-parametric:

$$F(x,y) = y^2(1-x) - x^2(1+x)$$

$$\nabla F = \left(-y^2 - 2x - 3x^2,\ 2y(1-x)\right)$$

$$\nabla F(0,0) = (0,0)$$

So the function is not guaranteed to be the graph of either $y=f(x)$ or $x=f(y)$.

Parametric: We see that if $t>0$ is such that $f(t)=(0,0)$, then

$$\frac{(-t)^2-1}{(-t)^2+1} = \frac{t^2-1}{t^2+1} = 0 \ \& \ (-t)\frac{(-t)^2-1}{(-t)^2+1} = -\left(t\frac{t^2-1}{t^2+1}\right) = -0 = 0$$

and so $f(-t)=(0,0)$ and there are two points that map to 0.