Math 311 Homework #5 Solutions

§1#2. Let $f:[a,b]\to\mathbb{R}$ be an integrable function.
(a) Show that the graph of f in $\mathbb{R}^2$ has zero content.

Let $\varepsilon>0$.

Since $f\in\mathcal{R}[a,b]$ there is a $P\in\mathcal{P}[a,b]$ such that $U(P,f)-L(P,f)<\varepsilon$.

Let $m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$
$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}$.

Define $R_i=[x_{i-1},x_i]\times[m_i,M_i]$, and the graph $S=\{(x,f(x)): x\in[a,b]\}\subseteq\mathbb{R}^2$.

Clearly $S\subseteq\bigcup_{i=1}^{n}R_i$. Claim $\sum_{i=1}^{n}\Delta A_i<\varepsilon$. This follows from the estimate

$$\sum_{i=1}^{n}\Delta A_i=\sum_{i=1}^{n}(x_i-x_{i-1})(M_i-m_i)$$

$$=\sum_{i=1}^{n}M_i\Delta x_i-\sum_{i=1}^{n}m_i\Delta x_i$$

$$=U(P,f)-L(P,f)<\varepsilon.$$

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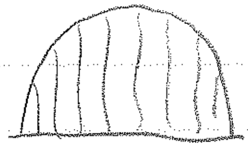
§4.3 #6 Evaluate the following double integrals.

(a).
$$\iint_S (x+3y^3)\,dA$$
where S is the upper half, $y \geq 0$, of the unit disk $x^2+y^2 \leq 1$.



$$\iint_S (x+3y^3)\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x+3y^3)\,dy\,dx$$

$$= \int_{-1}^{1} xy + \frac{3}{4}y^4 \Big|_0^{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} x\sqrt{1-x^2}\,dx + \frac{3}{4}(1-x^2)^2\,dx = I_1 + I_2$$

where

$$I_1 = \int_{-1}^{1} x\sqrt{1-x^2}\,dx = 0 \quad \text{by symmetry}$$

1a Continues....

$$I_2 = \frac{3}{4}\int_{-1}^{1} (1-x^2)^2\,dx$$

$$= \frac{3}{2}\int_0^1 (1-2x^2+x^4)\,dx$$

$$= \frac{3}{2}\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1$$

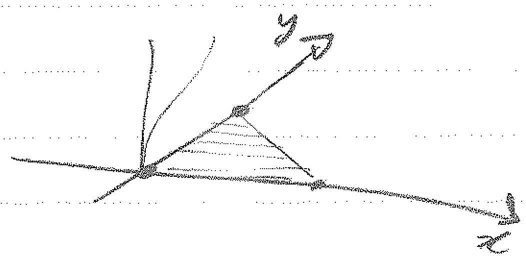
$$= \frac{3}{2}\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{3}{2}\left(\frac{15-10+3}{15}\right)$$

$$= \frac{3}{2}\,\frac{8}{15} = \frac{4}{5}$$

Thus

$$\iint_S (x^2+3y^2)\,dA = 0 + \frac{4}{5} = \frac{4}{5}$$

§9.3#2 Find the volume of the region above the triangle in the x-y plane with vertices $(0,0)$, $(1,0)$ and $(0,1)$ and below the surface $z = 6xy(1-x-y)$.



$$\int_0^1\int_0^{1-y} 6xy(1-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^{1-y} (6xy - 6x^2y - 6xy^2)\,dx\,dy$$

$$= \int_0^1 3x^2y - 2x^3y - 3x^2y^2 \Big|_0^{1-y} dy$$

$$= \int_0^1 3(1-y)^2y - 2(1-y)^3y - 3(1-y)^2y^2\,dy$$

$w = 1-y \qquad dw = -dy \qquad y = 1-w$

$$= \int_0^1 3w^2(1-w) - 2w^3(1-w) - 3w^2(1-w)^2\,dw$$

$$= \int_0^1 (3w^2 - 3w^3 - 2w^3 + 2w^4 - 3w^2 + 6w^3 - 3w^4)\,dw$$

$$= \int_0^1 w^3 - w^4\,dw = \frac{1}{4}w^4 - \frac{1}{5}w^5\Big|_0^1 = \frac{1}{4} - \frac{1}{5}$$

$$= \frac{5-4}{20} = \frac{1}{20}$$

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§4.3#5 Evaluate the following iterated integrals.

(a) $\int_1^3 \int_1^y y e^{2x} \, dx \, dy$

$= \int_1^3 y \frac{1}{2} e^{2x} \Big|_1^y dy$

$= \int_1^3 y \frac{1}{2} (e^{2y} - e^2) dy$

$= \int_1^3 \frac{1}{2} y e^{2y} dy - \int_1^3 \frac{1}{2} e^2 y \, dy = I_1 - I_2$

where

$I_1 = \int_1^3 \frac{1}{2} y e^{2y} dy = \frac{1}{4} \int_1^3 y \, de^{2y}$

$= \frac{1}{4} \left(y e^{2y} \Big|_1^3 - \int_1^3 e^{2y} dy \right)$

$= \frac{1}{4} \left(3e^6 - e^2 - \frac{1}{2} e^{2y} \Big|_1^3 \right)$

$= \frac{1}{4} \left(3e^6 - e^2 - \frac{1}{2} e^6 + \frac{1}{2} e^2 \right)$

$= \frac{1}{4} \left(\frac{5}{2} e^6 - \frac{1}{2} e^2 \right) = \frac{1}{8} (5e^6 - e^2)$

§4.3#5 continues...

and

$$I_2 = \int_1^3 \frac{1}{2} e^2 y\,dy = \frac{1}{2}e^2\left(\frac{1}{2}y^2\Big|_1^3\right)$$

$$= \frac{1}{4}e^2(9-1) = 2e^2$$

Thus

$$\int_1^3\int_1^y ye^{2x}\,dx\,dy = \frac{1}{8}(5e^6-e^2) - 2e^2$$

$$= \frac{1}{8}(5e^6-17e^2)$$

Doesn't agree with back of book, but is correct check with Maple...

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> restart;
> f:=int(y*exp(2*x),x=1..y);
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$$f := -\frac{1}{2}e^2 y + \frac{1}{2} y e^{(2y)}$$

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> int(f,y=1..3);
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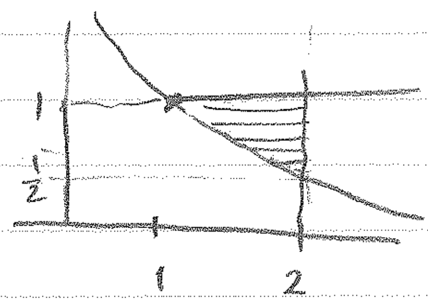
$$-\frac{17}{8}e^2 + \frac{5}{8}e^6$$

§4.3 #5(b) $\int_1^2 \int_{1/x}^1 y e^{xy}\, dy\, dx$

Try reversing order of integration:



$$\int_{1/2}^1 \int_{1/y}^2 y e^{xy}\, dx\, dy$$

$$= \int_{1/2}^1 \left(e^{xy} \Big|_{1/y}^2 \right) dy = \int_{1/2}^1 (e^{2y} - e)\, dy$$

$$= \frac{1}{2} e^{2y} \Big|_{1/2}^1 - ey \Big|_{1/2}^1$$

$$= \frac{1}{2}(e^2 - e) - e + \frac{1}{2} e$$

$$= \frac{1}{2} e^2 - e.$$

Prove or disprove: Let S be a closed bounded, Jordan measurable subset of $\mathbb{R}^n$, and let f be continuous and g be Riemann integrable on S. Then fg is Riemann integrable on S.

True. Proof:

Since f is continuous on a closed and bounded set S, ~~then~~ f is bounded and uniformly continuous. Let B be a bound on f.

Since g is Riemann integable it is bounded. Let C be a bound on g.

Let $\varepsilon > 0$, and $\varepsilon_1 = \varepsilon / 2B$.

Let R be a rectangle enclosing S with area ΔA. Thus, $g\chi_S$ is Riemann integrable on R. Choose a partition $P \in \mathcal{P}(R)$ such that

$$U(P, g\chi_S) - L(P, g\chi_S) < \varepsilon_1$$

Let $\varepsilon_2 = \varepsilon / 2C\Delta A$.

Let $\delta > 0$ be so small that $|x-y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon_2$.

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Let P^* be a refinement of P such that $|P^*| < \delta$. Then

$$U(P^*, fg\chi_S) - L(P^*, fg\chi_S)$$

$$= \sum M^*_{jk} \Delta A^*_{jk} - \sum m^*_{jk} \Delta A^*_{jk}$$

$$= \sum (M^*_{jk} - m^*_{jk}) \Delta A^*_{jk}$$

where

$$M^*_{jk} = \sup\{(fg\chi_S)(w) : w \in R^*_{jk}\}$$

and

$$m^*_{jk} = \inf\{(fg\chi_S)(w) : w \in R^*_{jk}\}.$$

Now

$$M^*_{jk} - m^*_{jk} = \sup\{|(fg\chi_S)(w_1) - (fg\chi_S)(w_2)| : w_1, w_2 \in R^*_{jk}\}.$$

and for $w_1, w_2 \in R^*_{jk}$ estimate

$$|fg\chi_S(w_1) - fg\chi_S(w_2)| \leq$$

$$= |f(w_1) g\chi_S(w_1) - f(w_1) g\chi_S(w_2)| + |f(w_1) g\chi_S(w_2) - f(w_2) g\chi_S(w_2)|$$

$$\leq |f(w_1)| |g\chi_S(w_1) - g\chi_S(w_2)| + |f(w_1) - f(w_2)| |g\chi_S(w_2)|$$

$$\leq B |g\chi_S(w_1) - g\chi_S(w_2)| + \varepsilon_2 C.$$

$$|(f g \chi_s)(w_1) - (f g \chi_s)(w_2)|$$

$$\leq B|g\chi_s(w_1) - g\chi_s(w_2)| + \varepsilon_2 C$$

$$\leq B(M_{jk} - m_{jk}) + \varepsilon_2 C$$

where $M_{jk} = \sup\{ g\chi_s(w) : w \in R_{jk}^* \}$

$m_{jk} = \inf\{ g\chi_s(w) : w \in R_{jk}^* \}$.

Taking supremum on the left side over $w_1, w_2 \in R_{jk}^*$ yields

$$M_{jk}^* - m_{jk}^* \leq B(M_{jk} - m_{jk}) + \varepsilon_2 C.$$

Therefore

$$\Sigma (M_{jk}^* - m_{jk}^*) \Delta A_{jk}^*$$

$$\leq B \Sigma (M_{jk} - m_{jk}) \Delta A_{jk}^* + \varepsilon_2 C \Sigma \Delta A_{jk}^*$$

$$\leq B(U(P^*, g\chi_s) - L(P^*, g\chi_s)) + \varepsilon_2 C \Delta A.$$

$$< B(U(P, g\chi_s) - L(P, g\chi_s)) + \frac{\varepsilon}{2}$$

$$< B\varepsilon_1 + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus there is a partition $P^* \in \mathcal{P}(R)$ such that

$$U(P^*, fg\chi_S) - L(P^*, fg\chi_S) < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$fg\chi_S \in \mathcal{R}(R)$$

or that fg is Riemann integrable on S.