Math 330 Final Exam Version A

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}.$$

Write A as LDU where L is lower triangular with ones on its diagonal, D is diagonal and U is upper triangular with ones on its diagonal.

2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Find the reduced row echelon form R of A.

3. Consider the matrix A with reduced row echelon form R given by

$$A = \begin{bmatrix} 6 & 0 & 6 & 1 & 4 \\ 0 & 3 & 6 & 3 & 0 \\ 1 & 0 & 1 & 0 & 5 \\ 1 & 0 & 1 & 4 & 5 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(i) Find a basis for the subspace $\mathcal{C}(A)$ and state its dimension.

(ii) Find a basis for the subspace $\mathcal{N}(A)$ and state its dimension.

4. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

5. Let

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2\sqrt{2}}{3} & \frac{-1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Note that Q is orthogonal and R upper triangular. Suppose $A = QR$. Find the x which minimizes $\|Ax - b\|$.

6. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Find $\det A$.

7. Let A and B be 3×3 matrices with A positive definite. Suppose $\det A = 3$ and $\det B = -1$.

(i) Find $\det(-2B)$.

(ii) Find $\det(B^T)$.

(iii) Find $\det(A^{1/2})$.

(iv) Find $\det(A^{-2}B)$.

8. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find the eigenvectors and eigenvalues of A

9. Let A be a matrix with eigenvalue-eigenvector pairs given by

λ	x
6	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
7	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
4	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Find $\sqrt{A}$.

10. Let

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}.$$

Find the singular value decomposition $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal.

Linear Algebra Formula Sheet

Orthogonal Projection

$$P = A(A^T A)^{-1} A^T$$

Least Squares

$$x = (A^T A)^{-1} A^T b$$

Gram-Schmidt

$$\begin{aligned}
\tilde{q}_1 &= a_1 & q_1 &= \tilde{q}_1 / \|\tilde{q}_1\| \\
\tilde{q}_2 &= a_2 - q_1(q_1 \cdot a_2) & q_2 &= \tilde{q}_2 / \|\tilde{q}_2\| \\
\tilde{q}_3 &= a_3 - q_1(q_1 \cdot a_3) - q_2(q_2 \cdot a_3) & q_3 &= \tilde{q}_3 / \|\tilde{q}_3\| \\
&\vdots & &\vdots \\
\tilde{q}_n &= a_n - q_1(q_1 \cdot a_n) - q_2(q_2 \cdot a_n) - \cdots - q_{n-1}(q_{n-1} \cdot a_n) & q_n &= \tilde{q}_n / \|\tilde{q}_n\|.
\end{aligned}$$

Least Squares using Gram-Schmidt

$$Rx = Q^T b \qquad \text{where} \qquad R = Q^T A$$

Combinatorial Determinant Formula

$$\det A = \sum_{\text{all } n \times n \text{ permuation matrices } P} (\det P)(\text{product of the diagonal of } PA)$$

Cofactor (Laplace's) Expansion

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{where} \qquad C_{ij} = (-1)^{i+j} \det M_{ij}$$

and M_{ij} is the submatrix of A resulting from deleting row i and column j.

Formula for the Inverse

$$A^{-1} = \frac{C^T}{\det(A)}$$

In class we defined C_{ij} with the meaning of the indices reversed. Thus, i corresponded to the deleted column and j to the deleted row in the minor. In this case, the transpose isn't necessary in the above formula. This also changes the cofactor expansion.

Cramer's Rule

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \cdots \qquad x_n = \frac{\det B_n}{\det A}$$