~Key~

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1. Determine whether the following pairs of vectors are perpendicular.

(i) Is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ perpendicular to $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$? No, not perpendicular.

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -1+1+1 = 1$$

(ii) Is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ penpendicular to $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$? YES, perpendicular.

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = -1+0+1 = 0$$

2. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

(i) Compute $\|v\|$.

$$= \sqrt{v \cdot v} = \sqrt{9+1+4} = \sqrt{14}$$

(ii) Compute Av.

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-2+8 \\ -6-3+2 \\ -12-1+4 \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \\ -9 \end{bmatrix},$$

3. Let $A \in \mathbf{R}^{m\times n}$ and $\mathcal{C}(A)$ be the columnspace of A. Then

(A) $\mathcal{C}(A) = \{ Ax : x \in R^n \}$.
(B) $\mathcal{C}(A) = \{ Ax : A \in R^m \}$.
(C) $\mathcal{C}(A) = \{ x \in \mathbf{R}^n : Ax = 0 \}$.
(D) $\mathcal{C}(A) = \{ x \in \mathbf{R}^m : Ax = 0 \}$.
(E) none of the above.

4. True or false with a counterexample if false and a reason if true.

(i) If $A \in \mathbf{R}^{n\times n}$ is invertible then A^2 is invertible.

True, the inverse of A^2 is $(A^{-1})^2$.

Check:

$$A^2(A^{-1})^2 = (AA)(A^{-1}A^{-1}) = A(AA^{-1})A^{-1}$$
$$= AIA^{-1} = AA^{-1} = I.$$

(ii) If P is a permutation matrix corresponding to the row operation $r_i \leftrightarrow r_j$ where $i \neq j$ then $\mathcal{N}(P)$ is trivial.

True, since $r_i \leftrightarrow r_j$ is invertible, then the function $f(x) = Px$ is one-to-one. Thus $f(x)=0$ implies $x=0$.
Therefore,

$$N(P) = \{x : Px = 0\} = \{x : f(x) = 0\} = \{0\}.$$

(iii) If R is the reduced row echelon form of A then $\mathcal{C}(A) = \mathcal{C}(R)$.

False. Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $r_2 \leftarrow r_2 - 2r_1$ gives that $R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Therefore

$$\mathcal{C}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \text{ and } \mathcal{C}(R) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

These are obviously different since all vectors in $\mathcal{C}(R)$ have second component equal 0.

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5. Let

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find E^{-1}.

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Let

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

Solve the system $Ax = b$. Find x.

$$[A|b] = \begin{bmatrix} 2 & 3 & 2 & 3 \\ 2 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix} \quad \begin{matrix} r_2 \leftarrow r_2 - 2r_1 \\ r_3 \leftarrow r_3 - 2r_3 \end{matrix} \quad \begin{bmatrix} 2 & 3 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$r_2 \leftrightarrow r_3 \begin{bmatrix} 2 & 3 & 2 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 3 \\ -x_2 &= -1 \\ x_3 &= 0 \end{aligned}$$

Back substitution

$x_3 = 0$

$x_2 = 1$

$$x_1 = \frac{3 - 3x_2 - 2x_3}{2} = \frac{3 - 3 - 0}{2} = 0$$

Solution is $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

7. Let $A \in \mathbf{R}^{4\times 7}$. Suppose A is a matrix that can be put into echelon form U using elimination without pivoting. How may row operations of the form $r_i \leftarrow r_i + \alpha r_j$ does the elimination algorithm take in general to put A into echelon form? Write these row operations in order.

In general there would be 6 row operations

$$r_2 \leftarrow r_2 + \alpha_1 r_1$$
$$r_3 \leftarrow r_3 + \alpha_2 r_1$$
$$r_4 \leftarrow r_4 + \alpha_3 r_1$$
$$r_3 \leftarrow r_3 + \alpha_4 r_2$$
$$r_4 \leftarrow r_4 + \alpha_5 r_2$$
$$r_4 \leftarrow r_4 + \alpha_6 r_3$$

Where the $\alpha_1, \alpha_2, \dots, \alpha_6$ are the real numbers needed for the elimination.

8. Let $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{l\times m}$. Prove that $(BA)^T = A^T B^T$.

$$((BA)^T)_{ij} = (BA)_{ji} = \sum_{k=1}^{m} (B)_{jk}(A)_{ki}$$

$$= \sum_{k=1}^{m} (B^T)_{kj}(A^T)_{ik}$$

$$= \sum_{k=1}^{m} (A^T)_{ik}(B^T)_{kj} = (A^T B^T)_{ij}$$

for $i = 1, \dots, n$ and $j = 1, \dots, l$.

Therefore $(BA)^T = A^T B^T$.

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9. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \cancel{4} \\ \cancel{4} \\ \cancel{8} \end{bmatrix} \begin{bmatrix} -6 \\ -12 \\ -18 \end{bmatrix}$$

The nullspace of A is

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} +1 \\ -2 \\ 1 \end{bmatrix} c : c \in \mathbf{R} \right\}.$$

One solution to $Ax = b$ is

$$x = \begin{bmatrix} -1 \\ -10 \\ 5 \end{bmatrix}.$$

Find all solutions.

Check that x is a solution $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ -10 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -12 \\ -18 \end{bmatrix}$.

The general solution is $x_p + x_n$ where $x_n \in N(A)$.
All solutions are given by

$$\left\{ \begin{bmatrix} -1 \\ -10 \\ 5 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{where } c \in \mathbb{R} \right\}$$

10. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}$$

Find the reduced row echelon form R of A.

$r_2 \leftarrow r_2 - 2r_1$ $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}$ $r_3 \leftarrow r_3 - 3r_2$ $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}$

$r_2 \leftarrow r_2 + \frac{2}{+11} r_3$
$r_1 \leftarrow r_1 - \frac{1}{11} r_3$ $\begin{bmatrix} 1 & 0 & 0 & 5/11 \\ 0 & 2 & 0 & 23/11 \\ 0 & 0 & 11 & -5 \end{bmatrix}$ $r_2 \leftarrow \frac{1}{2} r_2$
$r_3 \leftarrow \frac{1}{11} r_3$ $\begin{bmatrix} 1 & 0 & 0 & 5/11 \\ 0 & 1 & 0 & 23/22 \\ 0 & 0 & 1 & -5/11 \end{bmatrix}$

11. Let

$$A = \begin{bmatrix} 1 & 2 & 5 & 6 & 2 \\ 2 & 4 & 4 & 2 & 8 \\ 6 & 12 & 1 & 1 & -2 \\ 4 & 8 & -3 & -1 & -10 \end{bmatrix}.$$

The reduced row echelon form of A is

$$R = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ P & F & P & P & F \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 7/2 \\ 0 & 0 & 0 & 1 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

(i) Find $\dim(\mathcal{C}(A))$. $= 3$

Since there are 3 pivot variables.

(ii) Find $\dim(\mathcal{N}(A))$. $= 2$

Since there are 2 free variables.

(iii) Find a basis for $\mathcal{N}(A)$ and the nullspace matrix N corresponding to A.

$$N = \begin{bmatrix} -2 & 1/2 \\ 1 & 0 \\ 0 & -7/2 \\ 0 & 5/2 \\ 0 & 1 \end{bmatrix}$$

and a basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -7/2 \\ 5/2 \\ 1 \end{bmatrix} \right\}$.

12. Let $E_1, E_2 \in \mathbf{R}^{m\times m}$ be row operations of the form

$$E_1 = [r_i \leftarrow r_i + \alpha_1 r_j] \qquad \text{where} \qquad i \neq j$$

and

$$E_2 = [r_k \leftarrow r_k + \alpha_2 r_l] \qquad \text{where} \qquad k \neq l.$$

Is it always true that $E_1E_2 = E_2E_1$? If true explain why; if not provide a counter example where it is false.

False. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. Then

$$E_1E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$$ which are different.

13. Let

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 2 \\ 4 & 1 & 8 \end{bmatrix}.$$

Find a lower triangular matrix L and an upper triangular U such that $LU = A$.

$r_2 \leftarrow r_2 - 2r_1$
$r_3 \leftarrow r_3 - 4r_1$ $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 6 \\ 0 & 5 & 16 \end{bmatrix}$ $\quad r_3 \leftarrow r_3 - \frac{5}{3}r_2 \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & 6 \end{bmatrix}$

Therefore $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 5/3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & 6 \end{bmatrix}$.

Check work

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 5/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 2 \\ 4 & 1 & 8 \end{bmatrix}.$$