4) Consider the matrix A whose characteristic polynomial is given by

$$A = \begin{bmatrix} 7 & 12 & -3 \\ -5 & -8 & 1 \\ -4 & -6 & 0 \end{bmatrix} \quad \text{and} \quad p(\lambda) = \lambda^3 + \lambda^2 - 2\lambda$$

Find the eigenvalues of A and the corresponding eigenvectors.

First, factor $p(\lambda)$ and solve $p(\lambda)=0$ to find the eigenvalues:

$$\lambda^3 + \lambda^2 - 2\lambda = \lambda(\lambda^2 + \lambda - 2) = \lambda(\lambda+2)(\lambda-1) = 0$$

Therefore, the eigenvalues are

$$\lambda = 0, \quad \lambda = 1 \quad \text{and} \quad \lambda = -2.$$

Now, compute the null spaces $\mathrm{Nul}(A - \lambda I)$ to find the corresponding eigenvectors.

In each case $\dim \mathrm{Nul}(A - \lambda I) = 1$ so we can take the shortcut of only working with two linearly independent rows of the resulting matrices.

Case $\lambda = 0$. Then

$$A - \lambda I = \begin{bmatrix} 7 & 12 & -3 \\ -5 & -8 & 1 \\ -4 & -6 & 0 \end{bmatrix}$$

It appears the 2st and 3rd rows are the easiest to work with. Thus,

$$\begin{bmatrix} -5 & -8 & 1 \\ -4 & -6 & 0 \end{bmatrix} \quad r_2 \leftarrow \frac{1}{-6} r_2 \quad \begin{bmatrix} -5 & -8 & 1 \\ 2/3 & 1 & 0 \end{bmatrix}$$

$$r_1 \leftarrow r_1 + 8 r_2 \quad \begin{array}{c} \begin{array}{ccc} F & P & P \end{array} \\ \begin{bmatrix} 1/3 & 0 & 1 \\ 2/3 & 1 & 0 \end{bmatrix} \end{array}$$

which, though not exactly reduced row eschelon form, is just as good after switching columns.

$$\begin{aligned} x_3 &= -1/3\, x_1 \\ x_2 &= -2/3\, x_1 \end{aligned} \quad \text{so} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \\ -1/3 \end{bmatrix} x_1$$

Therefore the eigenvector is

$$\begin{bmatrix} 1 \\ -2/3 \\ -1/3 \end{bmatrix}$$

Case $\lambda = 1$. Then

$$A - \lambda I = \begin{bmatrix} 6 & 12 & -3 \\ -5 & -9 & 1 \\ -4 & -6 & -1 \end{bmatrix}$$

This time keep the first two rows

$$\begin{bmatrix} -5 & -9 & 1 \\ 6 & 12 & -3 \end{bmatrix} \quad r_2 \leftarrow r_2 + 3r_1 \quad \begin{bmatrix} -5 & -9 & 1 \\ -9 & -15 & 0 \end{bmatrix}$$

$$r_2 \leftarrow \frac{1}{-9} r_2 \begin{bmatrix} -5 & -9 & 1 \\ 1 & 5/3 & 0 \end{bmatrix} \quad r_1 \leftarrow r_1 + 5r_2 \quad \begin{array}{c} \begin{array}{ccc} P & F & P \end{array} \\ \begin{bmatrix} 0 & -2/3 & 1 \\ 1 & 5/3 & 0 \end{bmatrix} \end{array}$$

Thus $\quad x_1 = -5/3\, x_2 \quad\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 1 \\ 2/3 \end{bmatrix} x_2$

$\quad\quad\quad x_3 = 2/3\, x_2$

Therefore, the resulting eigenvector is

$$\begin{bmatrix} -5/3 \\ 1 \\ 2/3 \end{bmatrix}$$

Case $\lambda = -2$. Then

$$A - \lambda I = \begin{bmatrix} 9 & 12 & -3 \\ -5 & -6 & 1 \\ -4 & -6 & 2 \end{bmatrix}$$

Taking the last two rows, yields

$$\begin{bmatrix} -5 & -6 & 1 \\ -4 & -6 & 2 \end{bmatrix} \quad r_2 \leftarrow r_2 - 2r_1 \quad \begin{bmatrix} -5 & -6 & 1 \\ 6 & 6 & 0 \end{bmatrix}$$

$$r_2 \leftarrow \tfrac{1}{6} r_2 \begin{bmatrix} -5 & -6 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad r_1 \leftarrow r_1 + 5r_2 \quad \begin{array}{c} \begin{matrix} P & F & P \end{matrix} \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Thus,

$$x_1 = -x_2 \qquad \text{so} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_2$$
$$x_3 = x_2$$

Therefore, the resulting eigenvector is

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$