# Math 330 Linear Algebra Homework 12

**4.1** Do the following matrices have linearly independent columns?

(a) $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \\ 2 & -1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & -6 & -1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} -9 & 0 & 7 \\ 4 & 0 & -5 \\ -1 & 0 & 6 \end{bmatrix}$.

(d) $A = \begin{bmatrix} D \\ B \end{bmatrix}$, where $B$ is $m \times n$ and $D$ is a diagonal $n \times n$ matrix with nonzero diagonal elements.

(e) $A = ab^T$ where $a$ and $b$ are $n$-vectors and $n > 1$.

(f) $A = I - ab^T$ where $a$ and $b$ are $n$-vectors with $\|a\|\|b\| < 1$.

(a) By the Gram-Schmidt algorithm

$$\tilde{q}_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$$\tilde{q}_2 = \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}^T \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix} \frac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix} - \frac{1}{14}(-22) \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -1 \end{bmatrix} + \frac{11}{7} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/7 \\ -9/7 \\ 15/7 \end{bmatrix} \neq 0$$

Therefore the columns were linearly <u>independent</u>.

(b) Since the matrix has more columns than rows, then the dimension independence inequality implies the columns must be <u>dependent</u>.

#4.1 continues...

(c) The zero column is always dependent.

(d) Independent, because the diagonal matrix with nonzero elements implies that the only way a linear combination of vectors could be zero in the first $n$-elements is for all the coefficients in the combination to be zero.

(e) Dependent because by definition

$$A = ab^T = \begin{bmatrix} b_1 a & b_2 a & \cdots & b_n a \end{bmatrix}$$

which shows that each column is a multiple of $a$ and therefore dependent on one of the other columns.

(f) Independent. Suppose $Ax = 0$. Then

$(I - ab^T)x = 0$ implies $x = ab^T x$. But

then $\|x\| = \|ab^T x\| \leq \|a\| \|b\| \|x\|$ implies

$$(1 - \|a\|\|b\|)\|x\| \leq 0$$

Since $\|a\|\|b\| < 1$ this implies $1 - \|a\|\|b\| > 0$ and consequently that $\|x\| = 0$. This implies that $x = 0$. Since the only solution to $Ax = 0$ is $x = 0$, then the columns of $A$ are independent.

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**4.2** Suppose $A$ is a nonsingular $n \times n$ matrix, $u$ and $v$ are $n$-vectors, and $v^TA^{-1}u \neq -1$. Show that $A + uv^T$ is nonsingular with inverse

$$(A+uv^T)^{-1} = A^{-1} - \frac{1}{1+v^TA^{-1}u}A^{-1}uv^TA^{-1}.$$

Since $v^TA'v \neq -1$ then $\frac{1}{1+v^TA'u}$ makes sense. Thus the matrix

$$B = A^{-1} - \frac{1}{1+v^TA^{-1}u}A^{-1}uv^TA^{-1} \in \mathbb{R}^{n\times n}$$

makes sense and can be defined. Since it is dimensionally compatible with $A+uv^T$ and square, the only thing left is to verify that $B(A+uv^T) = I$.

Compute as:

$$\left(A^{-1} - \frac{1}{1+v^TA^{-1}u}A^{-1}uv^TA^{-1}\right)\left(A+uv^T\right)$$

$$= I - \frac{1}{1+v^TA^{-1}u}A^{-1}uv^T + A^{-1}uv^T - \frac{1}{1+v^TA^{-1}u}A^{-1}u(v^TA^{-1}u)v^T$$

$$= I - \frac{(\cancel{1+v^TA^{-1}u})(A^{-1}uv^T)}{\cancel{1+v^TA^{-1}u}} + A^{-1}uv^T$$

$$= I - \cancel{A^{-1}uv^T} + \cancel{A^{-1}uv^T} = I$$

Therefore

$$(A+uv^T)^{-1} = A^{-1} - \frac{1}{1+v^TA^{-1}u}A^{-1}uv^TA^{-1}.$$

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**4.8** Suppose $A$ is an $m \times p$ matrix with linearly independent columns and $B$ is a $p \times n$ matrix with linearly independent rows. We are not assuming that $A$ or $B$ are square. Define

$$X = AB, \qquad Y = B^\dagger A^\dagger$$

where $A^\dagger$ and $B^\dagger$ are the pseudo-inverses of $A$ and $B$. Show the following properties.

(a) $YX$ is symmetric.

(b) $XY$ is symmetric.

(c) $YXY = Y$.

(d) $XYX = X$.

Carefully explain your answers.

Since $A$ has linearly independent columns then $m \geq p$ and

$$A^\dagger = (A^TA)^{-1}A^T$$

Since $B$ has linearly independent rows then $p \leq n$ and

$$B^\dagger = B^T(BB^T)^{-1}.$$

(a) To see $YX$ is symmetric take transposes

$$(YX)^T = X^TY^T = (AB)^T(B^\dagger A^\dagger)^T = B^TA^T(A^\dagger)^T(B^\dagger)^T$$

$$= B^TA^T((A^TA)^{-1}A^T)^T(B^T(BB^T)^{-1})^T$$

$$= B^TA^TA((A^TA)^{-1})^T((BB^T)^{-1})^TB$$

$$= B^T\cancel{A^TA}\,\cancel{(A^TA)^{-1}}(BB^T)^{-1}B = B^T(BB^T)^{-1}B$$

on the other hand

$$YX = B^\dagger A^\dagger AB = B^T(BB^T)^{-1}\cancel{(A^TA)^{-1}A^TA}B$$

$$= B^T(BB^T)^{-1}B$$

So $(YX)^T = YX$. This means $YX$ is symmetric.

4.8 continues ...

(b) To see $XY$ is symmetric again take transposes

$$(XY)^T = Y^T X^T = (B^\dagger A^\dagger)^T (AB)^T = (A^\dagger)^T (B^\dagger)^T B^T A^T$$

$$= ((A^TA)^{-1}A^T)^T (B^T(BB^T)^{-1})^T B^T A^T$$

$$= A(A^TA)^{-1} \cancel{(BB^T)^{-1} B B^T} A^T = A(A^TA)^{-1}A^T.$$

on the other hand

$$XY = ABB^\dagger A^\dagger = A \cancel{B B^T (BB^T)^{-1}} (A^TA)^{-1} A^T$$

$$= A(A^TA)^{-1}A^T$$

Therefore $(XY)^T = XY$ and we see $XY$ is symmetric.

(c) Show $YXY = Y$.

$$YXY = B^\dagger \cancel{A^\dagger A} \cancel{B B^\dagger} A^\dagger = B^\dagger A^\dagger = Y$$

since $A^\dagger A = (A^TA)^{-1}A^TA = I$ and $BB^\dagger = BB^T(BB^T)^{-1} = I$

(d) Show $XYX = X$.

$$XYX = A \cancel{BB^\dagger} \cancel{A^\dagger A} B = AB = X.$$

again due to the fact that $A^\dagger$ is a left inverse for $A$ and $B^\dagger$ is a right inverse of $B$.

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**4.11** Consider the matrix $A = I + aJ$, where $a$ is a real scalar and $J$ is the reverser matrix

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

of size $n \times n$. We assume $n > 1$.

(a) For what values of $a$ is $A$ singular?

(b) Assuming $A$ is nonsingular, express its inverse as a linear combination of the matrices $I$ and $J$.

(a) Suppose $Ax = 0$. Then

$$(I + aJ)x = 0 \text{ or } x = -aJx$$

Since $\|Jx\|^2 = \sum_{i=1}^{n} |(Jx)_i|^2 = \sum_{i=1}^{n} |x_{n-i+1}|^2 = \sum_{i=1}^{n} |x_i|^2 = \|x\|^2$

Then

$$\|x\| = |a| \, \|x\| \text{ and } (1 - |a|)\|x\| = 0.$$

If $|a| \neq 1$ this implies $\|x\| = 0$ so $x = 0$ in which case $I + aJ$ has linearly independent columns and is invertible.

We now treat the cases $a = 1$ and $a = -1$.

Suppose $a = 1$ then take

$$x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} = e_1 - e_n$$

where $e_i$ are the standard basis elements of $\mathbb{R}^n$.

Then

$$(I + J)x = (I + J)(e_1 - e_n) = Ie_1 - Ie_n + Je_1 - Je_n$$

$$= \cancel{e_1} - \cancel{e_n} + \cancel{e_n} - \cancel{e_1} = 0$$

#4.11 continues...

Thus $x \neq 0$ but $Ax = 0$. This implies that $A$ is not invertible.

Suppose $a = -1$. Then take

$$x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = e_1 + e_n .$$

It follows that

$$(I - J)x = (I - J)(e_1 + e_n) = Ie_1 + Ie_n - Je_1 - Je_n$$
$$= \cancel{e_1} + \cancel{e_n} - \cancel{e_n} - \cancel{e_1} = 0$$

and again a non-zero $x$ has been found such that $Ax = 0$. This implies $A$ is not invertible.

In summary, if $|a| \neq 1$ then $I + aJ$ is invertible and if $|a| = 1$ then it is singular.

#4.11 continues...

(b) Express the inverse as a linear combination of the matrices $I$ and $J$.

First not that $I^2 = I$ and $J^2 = I$. Therefore a difference of squares yields the identity

$$(I - aJ)(I + aJ) = I^2 - a^2 J^2 = (1 - a^2) I$$

It follows since $|a| \neq 1$ that $1 - a^2 \neq 0$. Dividing then yields

$$\left(\frac{I - aJ}{1 - a^2}\right)(I + aJ) = I$$

consequently, as the matrices are square, we have

$$(I + aJ)^{-1} = \frac{I - aJ}{1 - a^2}.$$

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**7.4** Calculate the LU factorization without pivoting of the matrix

$$A = \begin{bmatrix} -3 & 2 & 0 & 3 \\ 6 & -6 & 0 & -12 \\ -3 & 6 & -1 & 16 \\ 12 & -14 & -2 & -25 \end{bmatrix}.$$

You can check your result in MATLAB, but you have to provide the details of your calculation.

How to solve this problem was not in the chapter, so it will be omitted in the grading. For completeness, here is how to solve the problem by hand.

$$\begin{bmatrix} -3 & 2 & 0 & 3 \\ 6 & -6 & 0 & -12 \\ -3 & 6 & -1 & 16 \\ 12 & -14 & -2 & -25 \end{bmatrix} \qquad \begin{aligned} r_2 &\leftarrow r_2 + 2r_1 \\ r_3 &\leftarrow r_3 - r_1 \\ r_4 &\leftarrow r_4 + 4r_1 \end{aligned}$$

$$\begin{bmatrix} -3 & 2 & 0 & 3 \\ 0 & -2 & 0 & -6 \\ 0 & 4 & -1 & 13 \\ 0 & -6 & -2 & -13 \end{bmatrix} \qquad \begin{aligned} r_3 &\leftarrow r_3 + 2r_2 \\ r_4 &\leftarrow r_4 - 3r_2 \end{aligned}$$

$$\begin{bmatrix} -3 & 2 & 0 & 3 \\ 0 & -2 & 0 & -6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 5 \end{bmatrix} \qquad r_4 \leftarrow r_4 - 2r_1$$

$$\begin{bmatrix} -3 & 2 & 0 & 3 \\ 0 & -2 & 0 & -6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -4 & 3 & 2 & 1 \end{bmatrix}$$

#7.4 <u>continues...</u>

Check the result

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 & 3 \\ 0 & -2 & 0 & -6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 & 3 \\ 6 & -6 & 0 & -12 \\ -3 & 6 & -1 & 16 \\ 12 & -14 & -2 & -25 \end{bmatrix}$$

$$\begin{array}{r} -12 \\ -18 \\ \hline \pm 30 \end{array} \qquad \begin{array}{r} 30 \\ -\ 5 \\ \hline 25 \end{array}$$