1. Let $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. Find an invertible matrix P such that $P^{-1}AP = \operatorname{diag}(1, 3)$ and hence prove that

$$A^n = \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I_2.$$

The eigenvalues of A are given by

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{vmatrix} = -\lambda(4-\lambda) + 3$$

$$= \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

Therefore $\lambda = 3$ and $\lambda = 1$.

The corresponding eigenvectors are

$\underline{\lambda = 1}$

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

implies $x_1 = x_2$ so $x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$

Taking $x_2 = 1$ for convenience then gives the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

#1 continues...

$\underline{\lambda = 3}$

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

implies $x_1 = 3x_2$ so $x = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix}$

Taking $x_2 = 1$ for convenience then gives the eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Now taking

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

yields that

$$AP = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = PD$$

#1 continues...

Consequently

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Now calculate P^{-1}.

$$P^{-1} = \frac{1}{\det P} \operatorname{adj} P$$

$$\det P = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2, \quad \operatorname{adj} P = \begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix}$$

Thus

$$P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Now

$$A = PDP^{-1}$$

$$A^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$$

$$\vdots$$

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^n \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}$$

#1 continues...

Multiply this out to obtain

$$A^n = \begin{bmatrix} 1 & 3^{n+1} \\ 1 & 3^n \end{bmatrix} \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 + (3/2)3^n & 3/2 - (3/2)3^n \\ -1/2 + (1/2)3^n & 3/2 - (1/2)3^n \end{bmatrix}$$ $\boxed{4}$

On the other hand

$$\frac{3^n-1}{2} A + \frac{3-3^n}{2} I_2 = \frac{3^n-1}{2}\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix} + \frac{3-3^n}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3^n-1}{2}4 + \frac{3-3^n}{2} & \frac{3^n-1}{2}(-3) \\ \frac{3^n-1}{2} & \frac{3-3^n}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4\cdot 3^n - 4 + 3 - 3^n}{2} & \frac{3-3^{n+1}}{2} \\ \frac{3^n-1}{2} & \frac{3-3^n}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}\cdot 3^n & \frac{3}{2} - \frac{3}{2}\cdot 3^n \\ -\frac{1}{2} + \frac{1}{2}3^n & \frac{3}{2} - \left(\frac{1}{2}\right)3^n \end{bmatrix}$$ $\boxed{4}$

Same!

which is the same.

2. If $A = \begin{bmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{bmatrix}$, prove that A^n tends to a limiting matrix

$$\begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

as $n \to \infty$.

Find the eigenvalues and eigenvectors of the matrix A. First note

$$\det(A - \lambda I) = \begin{vmatrix} 0.6-\lambda & 0.8 \\ 0.4 & 0.2-\lambda \end{vmatrix}$$

$$= (0.6-\lambda)(0.2-\lambda) - (0.4)(0.8)$$

For convenience let $\alpha = 10\lambda$. Then

$$= \frac{1}{100}\left[(6-\alpha)(2-\alpha) - 32\right] = 0$$

implies

$$\alpha^2 - 8\alpha - 20 = 0$$

$$(\alpha - 10)(\alpha + 2) = 0$$

so $\alpha = 10$ and $\alpha = -2$.

#2 continues...

Find the corresponding eigenvectors...

$$(A-\lambda I)x = 0$$

is equivalent to

$$(10A - \alpha I)x = 0$$

Therefore

$\alpha = -2$

$$\begin{bmatrix} 6+2 & 8 \\ 4 & 2+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

implies $x_1 = -x_2$ or $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\alpha = 10$

$$\begin{bmatrix} 6-10 & 8 \\ 4 & 2-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

implies $4x_1 = 8x_2$ or $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Consequently $P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

#2 continues...

and we have

$$\det P = \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = -1-2 = -3$$

$$P^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$$

Therefore

$$A^n = PD^nP^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -.2 & 0 \\ 0 & 1 \end{bmatrix}^n\left(\frac{1}{-3}\right)\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\left(\frac{1}{-3}\right)\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$= \frac{1}{-3}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$= \frac{1}{-3}\begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}.$$

as $n \to \infty$.

Math 330 Linear Algebra Homework 15

1. Consider the matrix

$$A = \begin{bmatrix} 7/2 & -1 \\ 1 & 1 \end{bmatrix}$$

(i) Find the polynomial $p(\lambda) = \det(A - \lambda I)$.

(ii) Solve $p(\lambda) = 0$ to find the eigenvalues λ_i of A.

(iii) For $i = 1, 2$ find nonzero solutions x to $(A - \lambda_i I)x = 0$.

(iv) Write down the solutions to the eigenvalue–eigenvector problem $Ax = \lambda x$.

(i) $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 7/2 - \lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$

$= \frac{1}{2} \begin{vmatrix} 7 - 2\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = \frac{1}{2}\left[(7-2\lambda)(1-\lambda) + 2\right]$

$= \frac{1}{2}\left[2\lambda^2 - 9\lambda + 9\right]$

$= \lambda^2 - \frac{9}{2} + \frac{9}{2}.$

(ii) $2\lambda^2 - 9\lambda + 9 = 0$

$(2\lambda - 3)(\lambda - 3) = 0$

Therefore $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = 3$.

#1 continues...

(iii)

$\lambda_1 = \frac{3}{2}$

$$\begin{bmatrix} 7/2 - 3/2 & -1 \\ 1 & 1 - 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Therefore $x_1 = \frac{1}{2} x_2$ and $x = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$

$$\begin{bmatrix} 1/2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Therefore $x_1 = 2x_2$ and $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(iv) Solutions to the eigenvector eigenvalue problem are

$\lambda = \frac{3}{2}$, $x = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ and $\lambda = 3$, $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Math 330 Linear Algebra Homework 15

2. Consider the matrix

$$A = \begin{bmatrix} -1 & -10 \\ -1/5 & 0 \end{bmatrix}$$

(i) Find the polynomial $p(\lambda) = \det(A - \lambda I)$.

(ii) Solve $p(\lambda) = 0$ to find the eigenvalues λ_i of A.

(iii) For $i = 1, 2$ find nonzero solutions x to $(A - \lambda_i I)x = 0$.

(iv) Write down the solutions to the eigenvalue–eigenvector problem $Ax = \lambda x$.

(i) $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -10 \\ -1/5 & -\lambda \end{vmatrix}$

$= (-1-\lambda)(-\lambda) - 2 = \lambda^2 + \lambda - 2$

(ii) $\lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

$\lambda_1 = -2$ and $\lambda_2 = 1$

(iii) $\underline{\underline{\lambda_1 = -2}}$

$\begin{bmatrix} 1 & -10 \\ -1/5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

so $x_1 = 10 x_2$ and $x = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$

$\underline{\underline{\lambda_2 = 1}}$

$\begin{bmatrix} -2 & -10 \\ -1/5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

So $x_1 = -5 x_2$ and $x = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

(iv) Solutions are

$\lambda = -2,\ x = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$ and $\lambda = 1,\ x = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

Math 330 Linear Algebra Homework 15

3. Consider the matrix

$$A = \begin{bmatrix} 5/3 & -2/3 & -2/3 \\ -1 & 1 & -1 \\ -1/3 & 4/3 & 10/3 \end{bmatrix}$$

(i) Find the polynomial $p(\lambda) = \det(A - \lambda I)$.

(ii) Solve $p(\lambda) = 0$ to find the eigenvalues λ_i of A.

(iii) For $i = 1, 2, 3$ find nonzero solutions x to $(A - \lambda_i I)x = 0$.

(iv) Write down the solutions to the eigenvalue–eigenvector problem $Ax = \lambda x$.

i) $\det(A - \lambda I) = \begin{vmatrix} 5/3 - \lambda & -2/3 & -2/3 \\ -1 & 1-\lambda & -1 \\ -1/3 & 4/3 & 10/3 - \lambda \end{vmatrix}$

$$= \frac{1}{9} \begin{vmatrix} 5-3\lambda & -2 & -2 \\ -1 & 1-\lambda & -1 \\ -1 & 4 & 10-3\lambda \end{vmatrix}$$

95

$$= \frac{1}{9}\Big[(5-3\lambda)(1-\lambda)(10-3\lambda) - 2 + 8 - 2(1-\lambda) + 4(5-3\lambda) - 2(10-3\lambda)\Big]$$

$$= \frac{1}{9}\Big[-9\lambda^3 + (15+9+30)\lambda^2 - \overset{-95}{(15+50+30)}\lambda + 56 - 2 + \cancel{20} - \cancel{20} + \underset{-4}{(2 - 12 + 6)}\lambda\Big]$$

$$= \frac{1}{9}\Big[-9\lambda^3 + 54\lambda^2 - 99\lambda + 54\Big]$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

#3 continues...

(ii) $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

check $\lambda = \pm 1, \pm 2, \pm 3, \pm 6$ by the rational root theorem

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

divided!

Thus

$$(\lambda-1)(\lambda^2-5\lambda+6) = 0$$

$$(\lambda-1)(\lambda-3)(\lambda-2) = 0$$

Therefore the eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = 2 \quad \text{and} \quad \lambda_3 = 3.$$

(iii) Equivalent to solving

$$\begin{bmatrix} 5-3\lambda & -2 & -2 \\ -1 & 1-\lambda & -1 \\ -1 & 4 & 10-3\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

by multiplying the first row by 3 and the third row by 3 as before.

3 continues...

(iii)

$\lambda = 1$

$$\begin{bmatrix} 2 & -2 & -2 \\ -1 & 0 & -1 \\ -4 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 .$$

these are the same equations. Solve by looking at the second equation first.

$$-x_1 - x_3 = 0 \qquad \text{so} \qquad x_1 = -x_3$$

Then

$$2x_1 - 2x_2 - 2x_3 = 0$$

$$x_1 - x_2 - x_3 = 0$$

$$x_2 = x_1 - x_3 = -2x_3$$

Therefore

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} x_3$$

Taking $x_3 = 0$ for convenience leads to the eigenvector $x = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

#3 continues...

$\lambda_2 = 2$

$$\begin{bmatrix} -1 & -2 & -2 \\ -1 & -1 & -1 \\ -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or equivalently

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ x_1 - 4x_2 - 4x_3 &= 0 \end{aligned}$$

multiply the first equation by two and add to the last

$$\begin{aligned} 2x_1 + 4x_2 + 4x_3 &= 0 \\ x_1 - 4x_2 - 4x_3 &= 0 \\ \hline 3x_1 &= 0 \end{aligned}$$

so $x_1 = 0$

Then equation 2 implies

$$x_2 + x_3 = 0$$

so $x_2 = -x_3$

Thus

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} x_3$$

Taking $x_3 = 1$ for convience gives the eigenvector

$$x = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

#3 continues...

$\lambda_3 = 3$

$$\begin{bmatrix} -4 & -2 & -2 \\ -1 & -2 & -1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

equivalently

$$2x_1 + x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 4x_2 - x_3 = 0$$

multiply the second equation by 2 and add to the last

$$2x_1 + 4x_2 + 2x_3 = 0$$
$$x_1 - 4x_2 - x_3 = 0$$
$$\overline{3x_1 + x_3 = 0} \qquad x_1 = -\tfrac{1}{3}x_3$$

substitute into first equation

$$2\left(-\tfrac{1}{3}x_3\right) + x_2 + x_3 = 0$$

$$x_2 = \tfrac{2}{3}x_3 - x_3 = -\tfrac{1}{3}x_3$$

Therefore

$$x = \begin{bmatrix} -\tfrac{1}{3}x_3 \\ -\tfrac{1}{3}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{3} \\ -\tfrac{1}{3} \\ 1 \end{bmatrix} x_3$$

Set $x_3 = 3$ for convenience. Then the eigenvector is

$$x = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

#3 continues...

(iv) The solutions to $Ax = \lambda x$ are

λ	x
1	$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$
2	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
3	$\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

Math 330 Linear Algebra Homework 15

4. It is known that the matrix A has eigenvalues

$$\lambda_1 = -2, \qquad \lambda_2 = 1 \qquad \text{and} \qquad \lambda_3 = 5$$

with corresponding eigenvectors

$$x_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad x_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

Find the matrix A.

$$P = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Then

$$A = PDP^{-1}$$

$$= \begin{bmatrix} 0 & -2 & 2 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$$

I did the computation to find P^{-1} and multiply everything out using Julia. See the next page

$$A = \begin{bmatrix} 3 & -4 & 8 \\ -5/2 & 0 & -4 \\ -3/4 & -3/2 & 1 \end{bmatrix}$$

math330hw15p4

December 10, 2020

Math 330 Homework 15 Problem 4

[2]:
```
using LinearAlgebra
```

[3]:
```
P=[0 -2 2; 2 1 -1; 1 1 0]
```

[3]:
```
3×3 Array{Int64,2}:
 0  -2   2
 2   1  -1
 1   1   0
```

[4]:
```
D=diagm([-2,1,5])
```

[4]:
```
3×3 Array{Int64,2}:
 -2  0  0
  0  1  0
  0  0  5
```

We compute A using the formula

$$A = PDP^{-1}$$

[5]:
```
A=P*D*inv(P)
```

[5]:
```
3×3 Array{Float64,2}:
  3.0   -4.0   8.0
 -2.5    0.0  -4.0
 -0.75  -1.5   1.0
```

[ ]:
```

```