Math 330 Linear Algebra Homework 1

1.5 *Interpreting sparsity.* Suppose the n-vector x is sparse, *i.e.*, has only a few nonzero entries. Give a short sentence or two explaining what this means in each of the following contexts.

(a) x represents the daily cash flow of some business over n days.

(b) x represents the annual dollar value purchases by a customer of n products or services.

(c) x represents a portfolio, say, the dollar value holdings of n stocks.

(d) x represents a bill of materials for a project, *i.e.*, the amounts of n materials needed.

(e) x represents a monochrome image, *i.e.*, the brightness values of n pixels.

(f) x is the daily rainfall in a location over one year.

(a) When x represents daily cash flow, sparsity means on most days there was no revenue.

(b) When x represents the annual dollar value of purchases of n products or services, sparsity means most products were not puechrased.

(c) When x represents a portfolio, sparsity means only a few diffent kinds of stock are held.

(d) when x a bill of materials, sparsity means most materials weren't on the bill.

(e) when x represents a monochrome image, sparsity means most of the image is black.

(f) when x is the daily rainfall in a location over one year, sparsity means you're in a desert.

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1.8 *Profit and sales vectors.* A company sells n different products or items. The n-vector p gives the profit, in dollars per unit, for each of the n items. (The entries of p are typically positive, but a few items might have negative entries. These items are called *loss leaders*, and are used to increase customer engagement in the hope that the customer will make other, profitable purchases.) The n-vector s gives the total sales of each of the items, over some period (such as a month), *i.e.*, s_i is the total number of units of item i sold. (These are also typically nonnegative, but negative entries can be used to reflect items that were purchased in a previous time period and returned in this one.) Express the total profit in terms of p and s using vector notation.

Total profit $= p^T s$.

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1.11 *Word count and word count histogram vectors.* Suppose the n-vector w is the word count vector associated with a document and a dictionary of n words. For simplicity we will assume that all words in the document appear in the dictionary.

(a) What is $\mathbf{1}^T w$?

(b) What does $w_{282} = 0$ mean?

(c) Let h be the n-vector that gives the histogram of the word counts, *i.e.*, h_i is the fraction of the words in the document that are word i. Use vector notation to express h in terms of w. (You can assume that the document contains at least one word.)

(a) The total number of words in the document is given by $\mathbf{1}^T w$.

(b) When $w_{282} = 0$ it means the 282^{nd} word in the dictionary is absent from the document.

(c) The fraction of word i given in the document is given by the histogram

$$h = \frac{w}{\mathbf{1}^T w}.$$

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1.16 *Inner product of nonnegative vectors.* A vector is called *nonnegative* if all its entries are nonnegative.

(a) Explain why the inner product of two nonnegative vectors is nonnegative.

(b) Suppose the inner product of two nonnegative vectors is zero. What can you say about them? Your answer should be in terms of their respective sparsity patterns, *i.e.*, which entries are zero and nonzero.

(a) The inner product of two non-negative vectors can be seen to be positive as follows:

Let $x, y \in \mathbb{R}^n$ be non-negative vectors. Then, by definition $x_i \geq 0$ and $y_i \geq 0$ for all $i = 1, \dots, n$. Since the product of non-negative numbers is again non-negative then $x_i y_i \geq 0$ for all $i = 1, \dots, n$. Finally, since the sum of nonnegative numbers is non-negative, then

$$x^T y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \geq 0.$$

Thus $x^T y$ is non-negative.

(b) Let $x, y \in \mathbb{R}^n$ be non-negative vectors. It follows that $x_i y_i \geq 0$ for $i = 1, \dots, n$. Therefore $x_i y_i = |x_i y_i|$ and $x^T y = \sum_{i=1}^{n} |x_i y_i|$. Since $x^T y = 0$ by assumption, then each of the terms in the sum must be zero. Thus $x_i y_i = 0$ for $i = 1, \dots, n$. In terms of sparsity patterns this means the only nonzero values of x lie in the sparsity pattern of y and vice versa.

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1.17 *Linear combinations of cash flows.* We consider cash flow vectors over T time periods, with a positive entry meaning a payment received, and negative meaning a payment made. A (unit) *single period loan*, at time period t, is the T-vector l_t that corresponds to a payment received of \$1 in period t and a payment made of $\$(1+r)$ in period $t+1$, with all other payments zero. Here $r > 0$ is the interest rate (over one period).

Let c be a \$1 $T-1$ period loan, starting at period 1. This means that \$1 is received in period 1, $\$(1+r)^{T-1}$ is paid in period T, and all other payments (*i.e.*, $c_2, \ldots, c_{T-1}$) are zero. Express c as a linear combination of single period loans.

First note that

$$l_1 = \begin{bmatrix} 1 \\ -(1+r) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 0 \\ 1 \\ -(1-r) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and}$$

in general that

$$l_t = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -(1-r) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow t\text{'th position}$$

on the other hand

$$c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -(1+r)^{T-1} \end{bmatrix}$$

In order to write c as

$$c = \beta_1 l_1 + \beta_2 l_2 + \cdots + \beta_{T-1} l_{T-1}$$

a linear combination of the one-period loans consider the following strategy.

1. Take out a loan for \$1 at $t=1$. Thus $\beta_1 = 1$.
2. To pay off the \$1 loan take out a new loan for $\$(1+r)$ at $t=2$. Thus $\beta_2 = 1+r$.
3. To pay off the $\$(1+r)$ loan take out a new loan for $\$(1+r)^2$ at $t=3$. Thus $\beta_3 = (1+r)^2$
4. Continue this process until $t = T-1$ to obtain $\beta_4 = (1+r)^3, \ldots, \beta_{T-1} = (1-r)^{T-2}$. This leaves a payment of $(1-r)^{T-1}$ due at $t=T$, same as the vector c.

Thus

$$c = l_1 + (1+r) l_2 + \cdots + (1+r)^{T-2} l_{T-1} = \sum_{k=1}^{T-1} (1+r)^{k-1} l_k .$$

expresses c as a linear combination of single period loans.