Math 330 Linear Algebra Homework 2

2.1 *Linear or not?* Determine whether each of the following scalar-valued functions of n-vectors is linear. If it is a linear function, give its inner product representation, *i.e.*, an n-vector a for which $f(x) = a^T x$ for all x. If it is not linear, give specific x, y, α, and β for which superposition fails, *i.e.*,

$$f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y).$$

(a) The spread of values of the vector, defined as $f(x) = \max_k x_k - \min_k x_k$.

(b) The difference of the last element and the first, $f(x) = x_n - x_1$.

(c) The median of an n-vector, where we will assume $n = 2k + 1$ is odd. The median of the vector x is defined as the $(k+1)$st largest number among the entries of x. For example, the median of $(-7.1, 3.2, -1.5)$ is -1.5.

(d) The average of the entries with odd indices, minus the average of the entries with even indices. You can assume that $n = 2k$ is even.

(e) Vector extrapolation, defined as $x_n + (x_n - x_{n-1})$, for $n \geq 2$. (This is a simple prediction of what x_{n+1} would be, based on a straight line drawn through x_n and x_{n-1}.)

(a) The function $f(x) = \max_k x_k - \min_k x_k$ is not linear since taking $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ yields

$$f(x+y) = f\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = 3 - 3 = 0$$

but

$$f(x) + f(y) = f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 2 - 1 + 2 - 1 = 2$$

(b) The function $f(x) = x_n - x_1$ is linear and can be written as $f(x) = a^T x$ where

$$a = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

(c) The median is not linear since taking

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix} \qquad \text{yields}$$

$$f(x+y) = f\left(\begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}\right) = 7$$

but

$$f(x)+f(y) = f\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) + f\left(\begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}\right) = 2+6 = 8$$

(d) The average of the entries with odd indices minus the average of the entries with even indices for n-vectors with $n=2k$ is linear and given by $f(x)=a^Tx$ where

$$a = \frac{1}{k}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{bmatrix} \quad \text{or} \quad a_i = \begin{cases} \frac{1}{k} & \text{if } i \text{ is odd} \\ -\frac{1}{k} & \text{if } i \text{ is even} \end{cases}$$

(e) The function $f(x) = x_n + (x_n - x_{n-1})$ where $n \geq 2$ is linear and given by $f(x) = a^Tx$ where

$$a = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2 \end{bmatrix} \quad \text{or} \quad a_i = \begin{cases} -1 & \text{if } i = n-1 \\ 2 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

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2.4 *Linear function?* The function $\phi : \mathbf{R}^3 \to \mathbf{R}$ satisfies

$$\phi(1,1,0) = -1, \qquad \phi(-1,1,1) = 1, \qquad \phi(1,-1,-1) = 1.$$

Choose one of the following, and justify your choice: ϕ must be linear; ϕ could be linear; ϕ cannot be linear.

The fuction $f(x)$ cannot be linear. If it was Then

$$f\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right)+f\left(\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right)=f\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right)=f(0)$$

and since all linear functions can be written in the form $f(x)=a^Tx$ then $f(0)=a^T0=0$.

On the other hand, by definition

$$f\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right)+f\left(\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right)=1+1=2\neq 0$$

Therefore, $f(x)$ could not be linear.

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2.10 *Regression model.* Consider the regression model $\hat{y} = x^T\beta + v$, where $\hat{y}$ is the predicted response, x is an 8-vector of features, β is an 8-vector of coefficients, and v is the offset term. Determine whether each of the following statements is true or false.

(a) If $\beta_3 > 0$ and $x_3 > 0$, then $\hat{y} \geq 0$.

(b) If $\beta_2 = 0$ then the prediction $\hat{y}$ does not depend on the second feature x_2.

(c) If $\beta_6 = -0.8$, then increasing x_6 (keeping all other x_is the same) will decrease $\hat{y}$.

(a) False, $\beta_3 > 0$ and $x_3 > 0$ don't necessarily imply that $\hat{y} \geq 0$. For example suppose $v < -x_3\beta_3$ and all the other entries of x were zero. Then

$$\hat{y} = x^T\beta + v = x_3\beta_3 + v < x_3\beta_3 - x_3\beta_3 = 0$$

(b) True. If $\beta_3 = 0$ then $x^T\beta$ doesn't depend on x_3 and consequently neither does $\hat{y}$.

(c) True. Suppose x_6 is increased to $x_6 + \varepsilon$ where $\varepsilon > 0$. Then the prediction changes from $x^T\beta + v$ to

$$x^T\beta + \varepsilon\beta_6 + v = x^T\beta - 0.8\varepsilon + v < x^T + v.$$

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2.12 *Price change to maximize profit.* A business sells n products, and is considering changing the price of *one* of the products to increase its total profits. A business analyst develops a regression model that (reasonably accurately) predicts the total profit when the product prices are changed, given by $\hat{P} = \beta^T x + P$, where the n-vector x denotes the fractional change in the product prices, $x_i = (p_i^{\text{new}} - p_i)/p_i$. Here P is the profit with the current prices, $\hat{P}$ is the predicted profit with the changed prices, p_i is the current (positive) price of product i, and p_i^{new} is the new price of product i.

(a) What does it mean if $\beta_3 < 0$? (And yes, this can occur.)

(b) Suppose that you are given permission to change the price of *one* product, by up to 1%, to increase total profit. Which product would you choose, and would you increase or decrease the price? By how much?

(c) Repeat part (b) assuming you are allowed to change the price of two products, each by up to 1%.

(a) If $\beta_3 < 0$ than increasing the price of product number 3 will decrease total profit.

(b) Note that an increase in profit will occur either when the price of a product where $\beta_i < 0$ is decreased of the price of a product where $\beta_i > 0$ is increased. In either case the amount of increase in profit is at most 1 percent of $|\beta_i|$. Therefore, chose a product i such that $|\beta_i|$ is maximal and decrease the price by 1 percent if $\beta_i < 0$ or increase it by 1 percent if $\beta_i > 0$.

(c) After chosing i and changing p_i as indicated in part (b), choose $j \neq i$ such that $|\beta_j|$ is maximal among the remaining products and change the price of product j accordingly.