Math 330 Linear Algebra Homework 3

3.2 *RMS value and average of block vectors.* **Let x be a block vector with two vector elements,** $x = (a, b)$, where a and b are vectors of size n and m, respectively.

(a) Express $\mathbf{rms}(x)$ in terms of $\mathbf{rms}(a)$, $\mathbf{rms}(b)$, m, and n.

(b) Express $\mathbf{avg}(x)$ in terms of $\mathbf{avg}(a)$, $\mathbf{avg}(b)$, m, and n.

(a) Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $x = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{n+m}$.

By definition

$$\text{rms}(x) = \frac{\|x\|}{\sqrt{n+m}}$$

where

$$\|x\| = \sqrt{x_1^2 + \cdots + x_{n+m}^2}$$

$$= \sqrt{a_1^2 + \cdots + a_n^2 + b_1^2 + \cdots + b_m^2}$$

Since

$$\|a\| = \sqrt{a_1^2 + \cdots a_n^2} \quad \text{and} \quad \|b\| = \sqrt{b_1^2 + \cdots + b_m^2}$$

Then $\|x\| = \sqrt{\|a\|^2 + \|b\|^2}$.

Furthermore, since

$$\text{rms}(a) = \frac{\|a\|}{\sqrt{n}} \quad \text{and} \quad \text{rms}(b) = \frac{\|b\|}{\sqrt{m}}$$

it follows that

$$\text{rms}(x) = \frac{\sqrt{n(\text{rms}(a))^2 + m(\text{rms}(b))^2}}{\sqrt{m+n}}.$$

(b) Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $x \in \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{n+m}$. By definition

$$\text{avg}(x) = \frac{x_1 + \cdots + x_{n+m}}{n+m} = \frac{a_1 + \cdots + a_n + b_1 + \cdots + b_m}{n+m}$$

Since

$$\text{avg}(a) = \frac{a_1 + \cdots + a_n}{n} \quad \text{and} \quad \text{avg}(b) = \frac{b_1 + \cdots + b_m}{m}$$

then

$$\text{avg}(x) = \frac{n\,\text{avg}(a) + m\,\text{avg}(b)}{n+m}.$$

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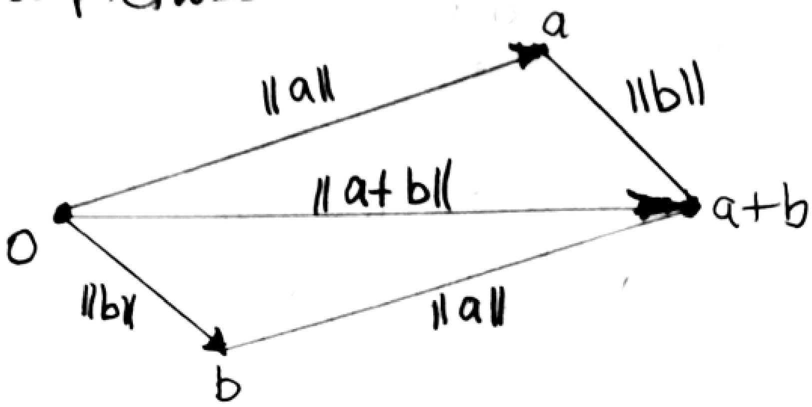
3.3 *Reverse triangle inequality.* Suppose a and b are vectors of the same size. The triangle inequality states that $\|a+b\| \leq \|a\| + \|b\|$. Show that we also have $\|a+b\| \geq \|a\| - \|b\|$. *Hints.* Draw a picture to get the idea. To show the inequality, apply the triangle inequality to $(a+b)+(-b)$.

The picture



contains two triangles. Since the length of one side of a triangle is always less than the sum of the lengths of the two other sides, then

$$\|a\| \leq \|b\| + \|a+b\|$$

$$\|a+b\| \leq \|a\| + \|b\|$$

$$\|b\| \leq \|a+b\| + \|a\|$$

The first of these inequalities can be re-arranged to obtain $\|a+b\| \leq \|a\| - \|b\|$.

algebraically, this may be seen as

$$\|a\| = \|(a+b)+(-b)\| \leq \|a+b\| + \|-b\|$$

Then using the fact that $\|-b\| = \|b\|$ and rearranging again yields $\|a+b\| \leq \|a\| - \|b\|$.

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3.6 *Taylor approximation of norm.* Find a general formula for the Taylor approximation of the function $f(x) = \|x\|$ near a given nonzero vector z. You can express the approximation in the form $\hat{f}(x) = a^T(x - z) + b$.

Recall equation (2.5) in the text which states

$$\hat{f}(x) = f(z) + \nabla f(z)^T (x - z).$$

Comparing this formula to the desired expression given that

$$a = \nabla f(z) \quad \text{and} \quad b = f(z).$$

Now, since $f(x) = \|x\|$ we obtain that

$$\nabla f(x) = \left(\frac{\partial}{\partial x_1} \|x\|, \ldots, \frac{\partial}{\partial x_n} \|x\| \right)$$

where

$$\frac{\partial}{\partial x_i} \|x\| = \frac{\partial}{\partial x_i} \sqrt{x_1^2 + \cdots + x_n^2} = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_n^2}}.$$

Therefore $\Delta f(x) = \frac{x}{\|x\|}$ and $a = \frac{z}{\|z\|}$.

Note also that $b = \|z\|$. It follows that

$$\hat{f}(x) = \frac{z^T(x - z)}{\|z\|} + \|z\|.$$

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3.8 *Converse Chebyshev inequality.* Show that at least one entry of a vector has absolute value at least as large as the RMS value of the vector.

Let $x \in \mathbb{R}^n$ and suppose

$$|x_i| < \mathrm{rms}(x) \quad \text{for } i = 1, \ldots, n,$$

that is, every entry of x had absolute value strictly less than $\mathrm{rms}(x)$. We show this leads to a contradiction and so couldn't happen, when then implies that at least one entry satisfies $|x_i| \geq \mathrm{rms}(x)$.

Now,

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} < \sqrt{n\,\mathrm{rms}(x)^2}$$

$$= \mathrm{rms}(x)\sqrt{n} = \frac{\|x\|}{\sqrt{n}}\sqrt{n} = \|x\|.$$

However $\|x\| < \|x\|$ is impossible. Therefore there must be at least one entry as large as the RMS value of the vector.

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3.7 *Chebyshev inequality.* Suppose x is a 100-vector with $\mathbf{rms}(x) = 1$. What is the maximum number of entries of x that can satisfy $|x_i| \geq 3$? If your answer is k, explain why no such vector can have $k + 1$ entries with absolute values at least 3, and give an example of a specific 100-vector that has RMS value 1, with k of its entries larger than 3 in absolute value.

Chebyshev's inequality states for $x \in \mathbb{R}^n$, if k of its entries satisfy $|x_i| \geq a$ where $a > 0$, then $k \leq \|x\|^2/a^2$.

Now, suppose x is a 100-vector with $\mathrm{rms}(x) = 1$. Then

$$\mathrm{rms}(x) = \frac{\|x\|}{\sqrt{n}} \quad \text{implies} \quad \|x\| = \sqrt{n} = 10$$

It follows, taking $a = 3$ in the Chebyshev inequality that
$$k \leq (10)^2/3^2 = 100/9 = 11.\overline{1}.$$

Thus, there are at most 11 entries in x with $|x_i| \geq 3$. Suppose 12 entries were such that $x_i = 3$. Then
$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2} \geq \sqrt{12 \cdot 9} = \sqrt{108}$$
which would imply that
$$\mathrm{rms}(x) = \frac{\sqrt{108}}{\sqrt{100}} \approx 1.04 > 1$$
contradicting the requirement that $\mathrm{rms}(x) = 1$.

For the example consider $x \in \mathbb{R}^{100}$ such that
$$x_i = \begin{cases} 3 + \varepsilon & \text{for } i \leq 11 \\ 0 & \text{for } i > 11 \end{cases}$$

We now solve for ε so that $\|x\| = 10$. By definition $\|x\| = \sqrt{11 \cdot (3+\varepsilon)^2} = (3+\varepsilon)\sqrt{11}$. Thus $(3+\varepsilon)\sqrt{11} = 10$ and $\varepsilon = \frac{10}{\sqrt{11}} - 3 \approx 0.015$.

It follows that the vector $x \in \mathbb{R}^{100}$ such that

$$x_i = \begin{cases} \frac{10}{\sqrt{11}} \approx 3.015 & \text{for } i \leq 11 \\ 0 & \text{for } i > 11 \end{cases}$$

has $\mathrm{rms}(x) = 1$ and 11 entries larger than 3.

Note, that there are many possible correct examples for x which also satisfy these properties.

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3.19 *Norm of sum.* Use the formulas (3.1) and (3.6) to show the following:

(a) $a \perp b$ if and only if $\|a+b\| = \sqrt{\|a\|^2 + \|b\|^2}$.

(b) Nonzero vectors a and b make an acute angle if and only if $\|a+b\| > \sqrt{\|a\|^2 + \|b\|^2}$.

(c) Nonzero vectors a and b make an obtuse angle if and only if $\|a+b\| < \sqrt{\|a\|^2 + \|b\|^2}$.

Draw a picture illustrating each case in 2-D.

Equation (3.1) and (3.6) are

$$\|x+y\| = \sqrt{\|x\|^2 + 2x^Ty + \|y\|^2}$$

and

$$\|x+y\|^2 = \|x\|^2 + 2\|x\|\,\|y\|\cos\theta + \|y\|^2$$

where θ is the angle between vectors x and y.

(a) $a \perp b$ means the angle between the vectors is $90°$ or equivalently $\pi/2$. Since

$$\cos\theta = \cos\pi/2 = 0$$

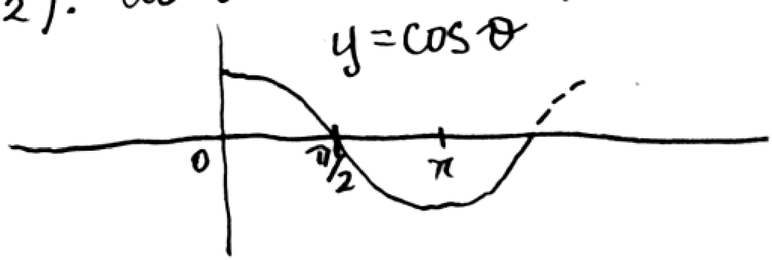
Then equation (3.6) implies

$$\|a+b\|^2 = \|a\|^2 + \cancel{2\|a\|\|b\|\cdot 0}^{\,0} + \|b\|^2$$

so taking square roots obtains

$$\|a+b\| = \sqrt{\|a\|^2 + \|b\|^2}$$

(b) The angle θ between a and be is acute if $\theta \in [0, \pi/2)$. As seen in the graph



it follows that $\cos\theta \in (0,1]$ when $\theta \in [0, \pi/2]$.

In particular $\cos\theta > 0$. Therefore, assuming that $\|a\| > 0$ and $\|b\| > 0$, which is clearly the case for there to even be an angle between them, then (3.6) implies

$$\|a+b\|^2 = \|a\|^2 + 2\|a\|\,\|b\|\cos\theta + \|b\|^2$$

$$> \|a\|^2 + 2\|a\|\,\|b\| \cdot 0 + \|b\|^2.$$

Consequently

$$\|a+b\| > \sqrt{\|a\|^2 + \|b\|^2}.$$

(c) If the angle between a and b is obtuse, then $\theta \in (\pi/2, \pi]$ and so $\cos\theta < 0$. In this case $\|a\|\,\|b\|\cos\theta < 0$ and equation (3.6) implies

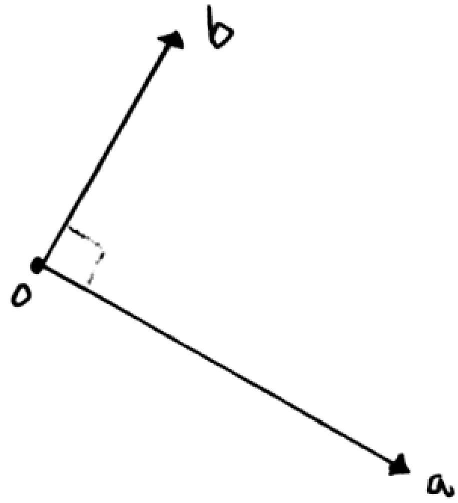
$$\|a+b\|^2 < \|a\|^2 + \|b\|^2$$

Consequently

$$\|a+b\| \leq \sqrt{\|a\|^2 + \|b\|^2}.$$

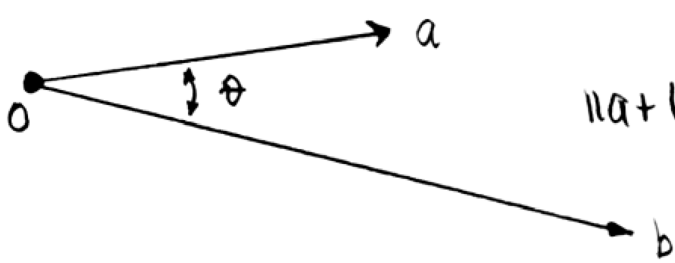
Pictures illustrating each of the three cases are

(a)



$$\|a+b\| = \sqrt{\|a\|^2 + \|b\|^2}$$

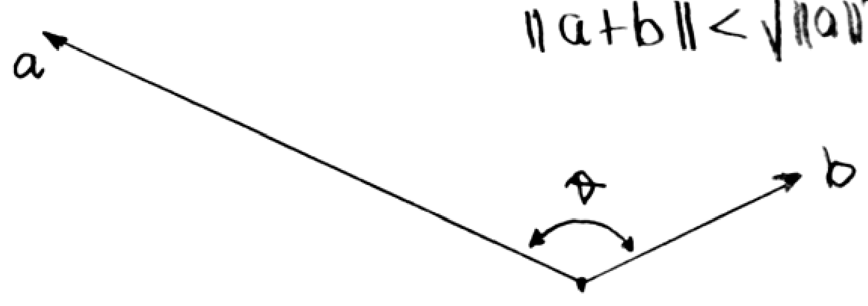
(b)



$$\|a+b\| > \sqrt{\|a\|^2 + \|b\|^2}$$

(c)

$$\|a+b\| < \sqrt{\|a\|^2 + \|b\|^2}$$



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3.25 *Leveraging.* Consider an asset with return time series over T periods given by the T-vector r. This asset has mean return μ and risk σ, which we assume is positive. We also consider cash as an asset, with return vector $\mu^{\mathrm{rf}}\mathbf{1}$, where μ^{rf} is the cash interest rate per period. Thus, we model cash as an asset with return μ^{rf} and zero risk. (The superscript in μ^{rf} stands for 'risk-free'.) We will create a simple portfolio consisting of the asset and cash. If we invest a fraction θ in the asset, and $1-\theta$ in cash, our portfolio return is given by the time series

$$p = \theta r + (1-\theta)\mu^{\mathrm{rf}}\mathbf{1}.$$

We interpret θ as the fraction of our portfolio we hold in the asset. We allow the choices $\theta > 1$, or $\theta < 0$. In the first case we are *borrowing* cash and using the proceeds to buy more of the asset, which is called *leveraging*. In the second case we are *shorting* the asset. When θ is between 0 and 1 we are blending our investment in the asset and cash, which is a form of *hedging*.

(a) Derive a formula for the return and risk of the portfolio, *i.e.*, the mean and standard deviation of p. These should be expressed in terms of μ, σ, μ^{rf}, and θ. Check your formulas for the special cases $\theta = 0$ and $\theta = 1$.

(b) Explain how to choose θ so the portfolio has a given target risk level σ^{tar} (which is positive). If there are multiple values of θ that give the target risk, choose the one that results in the highest portfolio return.

(c) Assume we choose the value of θ as in part (b). When do we use leverage? When do we short the asset? When do we hedge? Your answers should be in English.

(a) $$\operatorname{avg}(p) = \frac{p_1 + \cdots + p_T}{T} = \frac{\theta r_1 + \cdots + \theta r_2 + (1-\theta)T\mu^{rf}}{T}$$

$$= \theta \operatorname{avg}(r) + (1-\theta)\mu^{rf}.$$

$$\operatorname{std}(p) = \sqrt{\frac{(p_1 - \operatorname{avg}(p))^2 + \cdots + (p_T - \operatorname{avg}(p))^2}{T}}$$

$$= \sqrt{\frac{(\theta r_1 + (1-\theta)\mu^{rf} - \operatorname{avg}(p))^2 + \cdots + (\theta r_T + (1-\theta)\mu^{rf} - \operatorname{avg}(p))^2}{T}}$$

$$= \sqrt{\frac{(\theta r_1 - \operatorname{avg}(r))^2 + \cdots + (\theta r_T - \operatorname{avg}(r))^2}{T}} = |\theta| \operatorname{std}(r).$$

(b) To obtain $\text{std}(p) = \sigma^{tar}$ it is enough to solve $|\theta|\,\text{std}(r) = \sigma^{tar}$ for θ. In general there are two choices, one with $\theta > 0$ and one with $\theta < 0$. In particular, these are

$$\theta = \pm \frac{\sigma^{tar}}{\text{std}(r)}.$$

To maximize the portfolio return plug them in and compare values.

when $\theta = \dfrac{\sigma^{tar}}{\text{std}(r)}$ then

$$p = \frac{\sigma^{tar}}{\text{std}(r)} \text{avg}(r) + \left(1 - \frac{\sigma^{tar}}{\text{std}(r)}\right)\mu^{rf}$$

$$= \frac{\sigma^{tar}}{\text{std}(r)}\left(\text{avg}(r) - \mu^{rf}\right) + \mu^{rf}$$

when $\theta = -\dfrac{\sigma^{tar}}{\text{std}(r)}$ then

$$p = -\frac{\sigma^{tar}}{\text{std}(r)}\left(\text{avg}(r) - \mu^{rf}\right) + \mu^{rf}.$$

Which is greater depends on whether avg r or μ^{rf} is larger. Therefore, to maximize p choose

$$\theta = \begin{cases} \dfrac{\sigma^{tar}}{\text{std}(r)} & \text{if } \text{avg}(r) \geq \mu^{rf} \\[2ex] -\dfrac{\sigma^{tar}}{\text{std}(r)} & \text{if } \text{avg}(r) < \mu^{rf} \end{cases}.$$

(c) Choosing θ as in part (b) implies that we leverage when the mean return of the asset is greater than the cash interest rate and the target risk is greater than the risk of the asset.

We short the asset when average return of the asset is less than the cash interest rate.

We hedge when the mean return of the asset is more than the cash interest rate but the target risk is less than the risk of the asset.