Math 330 Linear Algebra Homework 4

1.1 *Average and norm.* Use the Cauchy–Schwarz inequality to prove that

$$-\frac{1}{\sqrt{n}}\|x\| \le \frac{1}{n}\sum_{i=1}^{n} x_i \le \frac{1}{\sqrt{n}}\|x\|$$

for all n-vectors x. In other words, $-\mathbf{rms}(x) \le \mathbf{avg}(x) \le \mathbf{rms}(x)$. What are the conditions on x to have equality in the upper bound? When do we have equality in the lower bound?

Recall the Cauchy-Schwarz inequality (page 57) states

$$|a^Tb| \le \|a\|\|b\|$$

with equality only when one vector is a multiple of the other.

Since $\sum_{i=1}^{n} x_i = \mathbb{1}^T x$ where $\mathbb{1} \in \mathbb{R}^n$ is the n-vector of all ones. Then

$$|\mathbb{1}^T x| \le \|\mathbb{1}\|\|x\| = \sqrt{n}\|x\|$$

In other words

$$\left|\sum_{i=1}^{n} x_i\right| \le \sqrt{n}\|x\|$$

Dividing by n yields

$$\frac{1}{n}\left|\sum_{i=1}^{n} x_i\right| \le \frac{\|x\|}{\sqrt{n}}$$

and solving for the absolute value, that

$$-\frac{\|x\|}{\sqrt{n}} \le \frac{1}{n}\sum_{i=1}^{n} x_i \le \frac{\|x\|}{\sqrt{n}}$$

To have equality in the upper bound the Cauchy-Schwarz to be an equality and the sum be non-negative.

The only way x could be a multiple of $\mathbb{1}$ is when x is a constant vector. Moreover, the constant must be non-negative. Thus,

$$x = \alpha \mathbb{1} \qquad \text{for some } \alpha \geqslant 0.$$

For equality in the lower bound

$$x = \alpha \mathbb{1} \qquad \text{for some } \alpha \leqslant 0$$

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1.2 Use the Cauchy–Schwarz inequality to prove that

$$\frac{1}{n}\sum_{k=1}^{n} x_k \geq \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right)^{-1}$$

for all n-vectors x with positive elements x_k. The left-hand side of the inequality is the arithmetic mean (average) of the numbers x_k; the right-hand side is called the harmonic mean.

Let $x \in \mathbb{R}^n$ such that $x_i > 0$ for $i = 1, \dots, n$.
Let $a \in \mathbb{R}$ be defined so $a_i = \sqrt{x_i}$ for $i = 1, \dots, n$
and $b \in \mathbb{R}$ be defined so $b_i = \frac{1}{\sqrt{x_i}}$ for $i = 1, \dots, n$.

Then

$$a^T b = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} \sqrt{x_i}\frac{1}{\sqrt{x_i}} = n.$$

while by the Cauchy-Schwantz inequality

$$|a^T b| \leq \|a\|\,\|b\| = \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$$

$$= \sqrt{\sum_{i=1}^{n}(\sqrt{x_i})^2}\sqrt{\sum_{i=1}^{n}\frac{1}{(\sqrt{x_i})^2}}$$

$$= \sqrt{\sum_{i=1}^{n} x_i} \cdot \sqrt{\sum_{i=1}^{n}\frac{1}{x_i}}$$

Putting these two things together yields

$$n \leq \sqrt{\sum_{i=1}^{n} x_i} \cdot \sqrt{\sum_{i=1}^{n} \frac{1}{x_i}}$$

or that

$$n^2 \leq \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)$$

Consequently

$$\frac{1}{n} \sum_{i=1}^{n} x_i \geq \frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}}$$

which was to be shown.