Math 330 Linear Algebra Homework 7

5.1 *Linear independence under combination.* Suppose $S = \{a, b, c\}$ and $T = \{d, e, f\}$ are two linearly independent sets of n-vectors. For each of the sets given below, determine which statement is correct. You may not use a computer to answer the questions. (Only one is correct in each case.)

(a) $\{a, b, c, d, e, f\}$

- is always linearly independent.
- is always linearly dependent.
- could be linearly independent or linearly dependent, depending on the values of $a, \ldots, f$.

(b) $\{a + d, b + e, c + f\}$

- is always linearly independent.
- is always linearly dependent.
- could be linearly independent or linearly dependent, depending on the values of $a, \ldots, f$.

(c) $\{a, a + b, a + b + c\}$

- is always linearly independent.
- is always linearly dependent.
- could be linearly independent or linearly dependent, depending on the values of $a, \ldots, f$.

(a) If $a = e_1, b = e_2, c = e_3, d = e_4, e = e_5, f = e_6$ they are all linearly independent. On the other hand if $a = e_1, b = e_2, c = e_3, d = 2e_1, e = 2e_2, f = 2e_3$ then $\{a, b, c\}$ and $\{d, e, f\}$ are both linearly independent but $\{a, b, c, d, e, f\}$ is not a linearly independent set.

(b) If $a = e_1, b = e_2, c = e_3, d = e_1, e = e_2, f = e_3$ then $\{a+d, b+c, c+f\} = \{2e_1, 2e_2, 2e_3\}$ is linearly independent. If instead $d = -e_1, e = -e_2, f = -e_3$ Then $\{a+d, b+c, c+f\} = \{0\}$. And the zero vector is never independent

(c) Suppose

$$\alpha_1 a + \alpha_2 (a+b) + \alpha_3 (a+b+c) = 0$$

To show that $\{a, a+b, a+b+c\}$ are linearly independent it is enough to show $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Collecting terms yields

$$(\alpha_1 + \alpha_2 + \alpha_3) a + (\alpha_2 + \alpha_3) b + \alpha_3 c = 0$$

Thus

$$\beta_1 a + \beta_2 b + \beta_3 c = 0$$

where $\beta_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\beta_2 = \alpha_2 + \alpha_3$ and $\beta_3 = \alpha_3$.

Since $\{a, b, c\}$ are linearly independent, it follows that $\beta_1 = \beta_2 = \beta_3 = 0$. From this we deduce $\alpha_3 = \beta_3 = 0$, $\alpha_2 = \beta_2 - \alpha_3 = 0$ and finally that $\alpha_1 = \beta_1 - \alpha_2 - \alpha_3 = 0 - 0 - 0 = 0$.

Therefore $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and the vectors $\{a, a+b, a+b+c\}$ are seen to be linearly independent.

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5.2 *Order of vectors in the Gram-Schmidt algorithm.* Suppose a_1, a_2 is a list of two linearly independent n-vectors. When we run the Gram-Schmidt algorithm on this list, we obtain the orthonormal vectors q_1, q_2.

Now suppose we run the Gram-Schmidt algorithm on the list of vectors a_2, a_1 (*i.e.*, the same vectors, in reverse order). Do we get the orthonormal vectors q_2, q_1 (*i.e.*, the orthonormal vectors obtained from the original list, in reverse order)?

If you believe this is true, give a very brief explanation why. If you believe it is not true, give a simple counter-example.

This is not true. Suppose

$$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then

$$q_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ so}$$

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Reversing the order of the vectors we have

$$q_2 = \frac{a_2}{\|a_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

from which it is clear the result will be different without even calculating the other vector

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5.2 *A surprising discovery.* An intern at a quantitative hedge fund examines the daily returns of around 400 stocks over one year (which has 250 trading days). She tells her supervisor that she has discovered that the returns of one of the stocks, Google (GOOG), can be expressed as a linear combination of the others, which include many stocks that are unrelated to Google (say, in a different type of business or sector).

Her supervisor then says: "It is overwhelmingly unlikely that a linear combination of the returns of unrelated companies can reproduce the daily return of GOOG. So you've made a mistake in your calculations."

Is the supervisor right? Did the intern make a mistake? Give a very brief explanation.

Let each vector $v_i \in \mathbb{R}^{250}$ represent the values of the ith stock with each element of that vector giving the value of the daily return for that day. Thus

$$(v_i)_j$$

represents the return of the ith stock on the jth day. Consider the set of vectors

$$\{ v_i \in \mathbb{R}^{250} : i = 1, \ldots, 400 \}.$$

describing the returns of all 400 stocks that were examined. By the Independence-dimension inequality this set of 400 vectors can not be linearly independent. Thus there exists $\alpha_i \in \mathbb{R}$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{400} v_{400} = 0$$

Suppose V_1 corresponds to the daily returns of GOOG. If $\alpha_1 \neq 0$ then it is possible to solve for V_1 as

$$V_1 = \frac{-\alpha_2}{\alpha_1} V_2 + \cdots + \frac{-\alpha_{400}}{\alpha_1} V_{400}$$

and it is clear GOOG can be written as a linear combination of the others. If $\alpha_1 = 0$, then it may or may not be possible. From a dimensional analysis point of view, however, chances are quite likely that its possible to find α's such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_{400} V_{400} = 0.$$

Thus, the intern did not make a mistake.

On the other hand, this kind of linear dependence is common in all big-data problems, so the manager thinking this implied there was some relationship between unrelated things was wrong.

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5.7 *Running Gram Schmidt algorithm twice.* We run the Gram Schmidt algorithm once on a given set of vectors $a_1, \ldots, a_k$ (we assume this is successful), which gives the vectors $q_1, \ldots, q_k$. Then we run the Gram–Schmidt algorithm on the vectors $q_1, \ldots, q_k$, which produces the vectors $z_1, \ldots, z_k$. What can you say about $z_1, \ldots, z_k$?

It will turn out that $z_i = q_i$ for $i = 1, \ldots, k$. This is because all the dot products that appear in the algorithm will be zero because the q's are orthogonal and then the renormalization step will do nothing because the $\tilde{z}$'s are already unit vectors.