Math 330 Linear Algebra Homework 9

2.1 *Projection on a line through the origin.* Let y be a nonzero n-vector, and consider the function $f : \mathbf{R}^n \to \mathbf{R}^n$, defined as

$$f(x) = \frac{x^T y}{\|y\|^2} y.$$

It can be shown that that $f(x)$ is the projection of x on the line passing through y and the origin (see exercise 3.12 of the textbook). Is f a linear function of x? If your answer is yes, give an $n \times n$ matrix A such that $f(x) = Ax$ for all x. If your answer is no, show with an example that f does not satisfy the definition of linearity $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$.

It is linear since

$$f(x) = \frac{(x^T y)}{\|y\|^2} y = y \frac{(x^T y)}{\|y\|^2} = y \frac{(y^T x)}{\|y\|^2}$$

$$= \frac{y y^T}{\|y\|^2} x = Ax$$

where

$$A = \frac{y y^T}{\|y\|^2}$$

Note that

$$y y^T = \begin{bmatrix} y_1 y_1 & y_1 y_2 & \cdots & y_1 y_n \\ y_2 y_1 & y_2 y_2 & \cdots & y_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n y_1 & y_n y_2 & \cdots & y_n y_n \end{bmatrix}$$

is the outer product of y with itself.

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2.5 *Circular convolution.* The circular convolution of two n-vectors a, b is the n-vector c defined as

$$c_k = \sum_{\substack{\text{all } i \text{ and } j \text{ with} \\ (i+j) \bmod n = \\ k+1}} a_i b_j, \qquad k = 1, \ldots, n,$$

where $(i+j) \bmod n$ is the remainder of $i+j$ after integer division by n. Therefore the sum is over all i, j with $i+j = k+1$ or $i+j = n+k+1$. For example, if $n = 4$,

$$\begin{aligned} c_1 &= a_1b_1 + a_4b_2 + a_3b_3 + a_2b_4 \\ c_2 &= a_2b_1 + a_1b_2 + a_4b_3 + a_3b_4 \\ c_3 &= a_3b_1 + a_2b_2 + a_1b_3 + a_4b_4 \\ c_4 &= a_4b_1 + a_3b_2 + a_2b_3 + a_1b_4. \end{aligned}$$

We use the notation $c = a \circledast b$ for circular convolution, to distinguish it from the standard convolution $c = a * b$ defined in the textbook (p. 136) and lecture (p. 3-32).

Suppose a is given. Show that $a \circledast b$ is a linear function of b, by giving a matrix $T_c(a)$ such that $a \circledast b = T_c(a)b$ for all b.

We first consider the case $n=4$ and obtain that

$$a \circledast b = \begin{bmatrix} a_1 & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_2 & a_1 & a_4 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Thus

$$T_c(a) = \begin{bmatrix} a_1 & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_2 & a_1 & a_4 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}$$

As the pattern is obvious, we can immidiately deduce that for arbitrary n

$$T_c(a) = \begin{bmatrix} a_1 & a_n & \cdots & & a_2 \\ a_2 & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & \ddots & & \ddots & a_n \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{bmatrix}.$$