Math 330: Quiz 1 Version A

This is a closed-book closed-notes quiz monitored through Zoom. Please enable both your web camera and your screen share during the quiz. You may send the instructor a private message to the instructor if you have a question or find an error in the quiz; otherwise, do not send messages in chat. Work each problem using pencil and paper on a clean sheet of paper. Be sure to write your name on each sheet of paper!

When you are finished use the raise hand feature of Zoom and I will move you to a breakout room where you can show me your student ID and completed work. Do not leave Zoom without first showing me your work in the breakout room. After you are done in the breakout room, please log out of Zoom and upload a high-resolution version of your work for grading to WebCampus. It is extremely important that you not make any changes in your answers before uploading them to WebCampus.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Suppose $u, v \in \mathbf{R}^3$ are given by $u = (7, -3, 1)$ and $v = (1, 2, -3)$.

(i) Find $2u - v$.

(ii) Find $u^T v$.

3. Suppose $x \in \mathbf{R}^4$ is given by $x = (1, 3, 5, -1)$.

(i) Find $\|x\|$.

(ii) Find $\mathbf{std}(x)$.

Note: Please do not use a calculator to approximate any square roots but leave your answer as an exact form.

4. Let $x = (a, b)$ be a block vector where $a \in \mathbf{R}^m$ and $b \in \mathbf{R}^n$. Express $\mathbf{avg}(x)$ in terms of $\mathbf{avg}(a)$, $\mathbf{avg}(b)$, m and n.

5. Recall the **Chebyshev inequality.** *Suppose x is an n-vector and k of its entries satisfy $|x_i| \geq a$ where $a > 0$. Then $k/n \leq (\mathbf{rms}(x)/a)^2$.*

Let $x \in R^{32}$ be such that $\|x\| = 10$. At most how many different entries of x could satisfy $|x_i| \geq 5$?

6. Explain the steps in the k-means algorithm.

7. [Extra Credit] Use the Cauchy–Schwarz inequality $|a^T b| \leq \|a\|\|b\|$ to prove the triangle inequality $\|a + b\| \leq \|a\| + \|b\|$.

1. I have worked independently on this quiz.
— Test Student.

2.(i) $2\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ -6 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 13 \\ -8 \\ 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 7 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 7-6-3 = -2$

3.(i) $\left\| \begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right\| = \sqrt{1+9+25+1} = \sqrt{36} = 6$

(ii) $\operatorname{avg}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right) = \frac{1+3+5-1}{4} = \frac{8}{4} = 2$

$\operatorname{std}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right) = \sqrt{\frac{(1-2)^2+(3-2)^2+(5-2)^2+(-1-2)^2}{4}}$

$= \frac{\sqrt{1+1+9+9}}{2} = \frac{\sqrt{20}}{2} = \sqrt{5}$

4. $x = \begin{bmatrix} a \\ b \end{bmatrix}$ where $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$.

$$\operatorname{avg}(x) = \frac{x_1 + \cdots + x_{m+n}}{m+n} = \frac{a_1 + \cdots + a_m + b_1 + \cdots + b_n}{m+n}$$

$$= \frac{m \operatorname{avg}(a) + n \operatorname{avg}(b)}{m+n}.$$

5. $x \in \mathbb{R}^{32}$ and $\|x\| = 10$. By Chebyshev if k of the entries of x satisfy $|x_i| \geq 5$, then

$$\frac{k}{32} \leq \left(\frac{\operatorname{rms}(x)}{5}\right)^2$$

By definition

$$\operatorname{rms}(x) = \frac{\|x\|}{\sqrt{32}} = \frac{10}{\sqrt{32}}$$

Thus

$$\frac{k}{32} \leq \frac{100}{32 \cdot 25}$$

implies

$$k \leq \frac{100}{25} = 4$$

Therefore, at most 4 entries of x could satisfy $|x_i| \geq 5$.

6. The k-means algorithm starts with vectors $x_1, x_2, \ldots, x_n$ and divides them into k groups.

The first step chooses $z_1, \ldots, z_k$ at random near the data given by the x's.

In the second step the data is divided into k groups $G_1, \ldots, G_k$ defined so that

$$i \in G_j \quad \text{if} \quad \|x_i - z_j\| \leq \min\{\|x_i - z_\ell\| : \ell = 1, \ldots, k\}$$

Next the points z_j are updated to be the centroids of each group as

$$z_j = \frac{1}{|G_j|} \sum_{i \in G_j} x_i$$

where $|G_j|$ represent the number of indices i's in the group G_j.

At this point the data is again divided into k-groups as in the second step. This process repeats until the members of the groups ceases to change, at which point the x's have been divided into k groups.

7. The Cauch-Schwarz inqualiity is

$$|a^T b| \leq \|a\| \|b\|$$

Therefore

$$\|a+b\|^2 = (a+b)^T (a+b)$$

$$= a^T a + a^T b + b^T a + b^T b$$

$$= \|a\|^2 + 2a^T b + \|b\|^2$$

$$\leq \|a\|^2 + 2\|a\| \|b\| + \|b\|^2$$

$$= (\|a\| + \|b\|)^2$$

implies

$$\|a+b\| \leq \|a\| + \|b\|.$$