A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \quad \cdots \quad 0 \quad b] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Gaussian Elimination

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

To use the algorithm Gaussian-Elimination to solve:

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

① Write the augmented matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 7 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$$

② Find the echelon form of $[A|b]$ by performing row operations

① $r_i \leftarrow r_i + \alpha r_j$ $\quad \alpha \neq 0 \quad i \neq j$ (Elimination)

② $r_i \leftarrow \alpha r_j$ $\quad \alpha \neq 0$ (Scaling)

③ $r_i \leftrightarrow r_j$ $\quad i \neq j$ (Swap)

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & & & \\ P & & & \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \end{array}$$

← pivot column ... where I make some zeros

turn 3 into a zero using elimination

$r_2 \leftarrow r_2 - 3r_1$

$9 - 3 \cdot 3 = 0$

$7 - 3 \cdot 4 = -5$

$6 - 3 \cdot 7 = -15$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

Check is there any row that is all zeros on the left of the red line and non-zero on the right?

No, so consistent and continue to step 3

Why does this check work?
Translate to alg notation ...

$$x_1 + 3x_2 + 4x_3 = 7$$
$$-5x_3 = -15$$

if that -5 were instead zero there'd be a contradiction

③ Make reduced row echelon form:

go backwards starting with this column

turn the 4 into a 0.

$$\begin{array}{cccc} & & x_3 & \\ & & P & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

coefficients of variables

constants on right side

But first rescale the last row...

$$r_2 \leftarrow -\frac{1}{5} r_2$$

$$\left[\begin{array}{cccc} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$r_1 \leftarrow r_1 - 4 r_2$$

$$\begin{array}{cccc} P & F & P & \\ x_1 & x_2 & x_3 & b \end{array}$$
$$\left[\begin{array}{cccc} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$1 - 4\cdot 0 = 1$$
$$3 - 4\cdot 0 = 3$$
$$4 - 4\cdot 1 = 0$$
$$7 - 4\cdot 3 = -5$$

this is now in reduced row echelon form

Note the column for x_2 was never considered for one of the elemination steps... it's called a free variable...

So I don't solve for it

④ Write in algebraic form

$$\begin{cases} x_1 + 3x_2 = -5 \\ x_3 = 3 \end{cases}$$

Solution

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 = x_2 \\ x_3 = 3 \end{cases}$$

where x_2 is a free variable... means you can take $x_2 =$ anything and get another solution...

Since there are free variables, then there are an infinite # of solutions!

Solution in vector form

Standard way to write solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 3x_2 \\ x_2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$