Linear function $f: \mathbb{R}^n \to \mathbb{R}^m$

① $f(u+v) = f(u) + f(v)$ for $u, v \in \mathbb{R}^n$

② $f(\alpha u) = \alpha f(u)$ for $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$

Example

$$f(x) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \boxed{2x_1 + 3x_2 + 4x_3} \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix} = Ax$$

where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$

Dot product: $x, y \in \mathbb{R}^n$ then $x \cdot y = \sum_{i=1}^{n} x_i y_i$

$$(1,2,3) \cdot (2,4,6) = 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 = 28$$

$$(2,3,4) \cdot (x_1, x_2, x_3) = 2x_1 + 3x_2 + 4x_3$$

$$\begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix} = \begin{bmatrix} (2,3,4) \cdot (x_1, x_2, x_3) \\ (-1,5,-3) \cdot (x_1, x_2, x_3) \\ (6,-2,8) \cdot (x_1, x_2, x_3) \end{bmatrix}$$

factored out the coefficients in the linear function f using a dot product

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$$

In general if $A = \begin{bmatrix} r_1 \\ \hline r_2 \\ \hline \vdots \\ \hline r_m \end{bmatrix}$

then

$$Ax = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix}$$

Matrix vector

inner product or dot prod. representation or row representation of the Matrix-vector product.

$$f(x) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}\Bigg|_{\substack{x_1 = 1 \\ x_2 = 0 \\ x_3 = 0}} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

e_1

also

$$f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} \qquad f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

e_2 e_3

The e_i's are called the standard basis vectors... They correspond to the x, y and z axes geometrically...

Note $$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} = \left[\, f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \,\middle|\, f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \,\middle|\, f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \,\right]$$

① . $f(u+v) = f(u) + f(v)$

② $f(\alpha u) = \alpha f(u)$

$$f\left(\begin{bmatrix}2\\0\\0\end{bmatrix}\right) = f\left(\underbrace{2}_{\alpha}\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{u}\right) = 2f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 2\begin{bmatrix}2\\-1\\6\end{bmatrix}$$

first column of matrix

$$f\left(\begin{bmatrix}2\\0\\1\end{bmatrix}\right) = f\left(\begin{bmatrix}2\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = f\left(\begin{bmatrix}2\\0\\0\end{bmatrix}\right) + f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= 2\begin{bmatrix}2\\-1\\6\end{bmatrix} + \begin{bmatrix}4\\-3\\8\end{bmatrix}$$

third column of matrix. . .

Thus

$$f(x) = f\left(\begin{bmatrix}x_1\\x_2\\x_2\end{bmatrix}\right) = x_1\begin{bmatrix}2\\-1\\6\end{bmatrix} + x_2\begin{bmatrix}3\\5\\-2\end{bmatrix} + x_3\begin{bmatrix}4\\-3\\8\end{bmatrix}.$$

column representation of the Matrix-vector product.

In general

$$Ax = \left[\begin{array}{c|c|c|c} c_1 & c_2 & \cdots & c_n \end{array}\right]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = x_1c_1 + x_2c_2 + \cdots + x_nc_n$$

Note this is related to the outer product or Inner Product

Dot product: $x, y \in \mathbb{R}^n$ then $x \cdot y = \sum_{i=1}^{n} x_i y_i$

$$(1,2,3)\cdot(2,4,6) = 1\cdot2 + 2\cdot4 + 3\cdot6 = 28$$

$$(2,3,4)\cdot(x_1,x_2,x_3) = 2x_1 + 3x_2 + 4x_3$$

Outer product

$$(1,2,3) \otimes (2,4,6) = \begin{bmatrix} 1\cdot2 & 1\cdot4 & 1\cdot6 \\ 2\cdot2 & 2\cdot4 & 2\cdot6 \\ 3\cdot2 & 3\cdot4 & 3\cdot6 \end{bmatrix}$$

not much more about outer products until column operations, least squares, approximations and singular value decomposition 7.2

The homogeneous equation $\boxed{A\mathbf{x} = \mathbf{0}}$ has a nontrivial solution if and only if th equation has at least one free variable.

Solve it the same as before, except easier because the right side is zero.

For $Ax=b$ write augmented matrix $[A|b]$ then do row operations to make echelon and reduced echelon form...

With $Ax = 0$ or $Ax = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Augmented matrix $\left[A \mid 0 \right]$

after every row operation the last column is still 0.

Example: $Ax = 0$

$$A = \begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix} \qquad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} -1 & 3 & -2 \\ 3 & -9 & 6 \end{bmatrix} \qquad r_2 \leftarrow r_2 + 3r_1$$

$$\begin{bmatrix} -1 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow -1\, r_1$$

reduced echelon form

$$\begin{array}{c} \begin{array}{ccc} P & F & F \end{array} \\ \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

read off solution

RHS is still 0.

Algebraic form

$$x_1 - 3x_2 + 2x_3 = 0$$

$$x_1 = 3x_2 - 2x_3$$

solve for pivot in terms of free

Answer

$$x = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$