§1.7 Linear independence

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$\boxed{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}}$$

← Right side of our equation... Homogeneous

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

Same equation

Idea: put the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ into a matrix.

$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ $\quad$ consider $\boxed{Ax = 0}$

$v_1 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$

$A = \begin{bmatrix} 5 & 7 & -2 \\ 1 & 2 & -1 \\ 0 & -6 & 6 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad$ consider $Ax = 0$

homogeneous system of linear equations.

Idea: Try to solve $Ax = 0$
(using row operations)

$\begin{bmatrix} 5 & 7 & -2 \\ 1 & 2 & -1 \\ 0 & -6 & 6 \end{bmatrix} \quad r_1 \leftrightarrow r_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 5 & 7 & -2 \\ 0 & -6 & 6 \end{bmatrix} \qquad r_2 \leftarrow r_2 - 5r_1$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix} \qquad r_3 \leftarrow r_3 - 2r_2$$

already zero by luck

So go on to the next column.

Echelon form

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_2 \leftarrow -\frac{1}{3} r_2$$

needs to be zero

$$\begin{array}{c} \begin{array}{ccc} P & P & F \end{array} \\ \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad r_1 \leftarrow r_1 - 2r_2$$

always same index here

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \\ P & P & F \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

solve for pivot variables in terms of the free vbls.

read off solution

$$x_1 + x_3 = 0 \qquad x_1 = -x_3$$
$$x_2 - x_3 = 0 \qquad x_2 = x_3$$

Answer in parametric form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

$Ax = 0$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

all solutions are $x = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ where x_3 is free.

Since there are lots of solutions then linearly dependent

Note that setting $x_3 = 1$ to get $c = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ as a solution such that $Ac = 0$. Thus,

$$-1v_1 + 1v_2 + 1v_3 = 0$$

$$-1\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} + 1\begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix} = 0$$

In this case I could have guessed the dependency relationship. Not always obvious...

Note since the solution $c \neq 0$ you can solve for one vector in terms of the others.

$$\underset{-1}{\cancel{-1}}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + \underset{-1}{\cancel{1}} \cdot \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} = \underset{-1}{\cancel{-1}}\begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}.$$

In general we have

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

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(8)

$$A = \begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$

solve $Ax = 0$

$r_2 \leftarrow r_2 + 3r_1$

$$\begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & -2 & 8 & -4 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$

~~$r_2 \leftrightarrow r_3$~~

ok. but do something else..

alternatively one could rescale as $r_2 \leftarrow -\frac{1}{2} r_2$

better

$$\begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & 1 & -4 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$

$r_3 \leftarrow r_3 - r_2$

Echelon form

$$\begin{array}{cccc} P & P & F & P \\ \end{array}$$
$$\begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Already know because there is a free variable that $Ax=0$ has lots of solutions.

Are the columns of A linearly independent? NO

Note every time a matrix is wider than it is tall there is guaranteed to be free variables... In this case the columns of A are always linearly dependent.