$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$$

Question are the vectors $\begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -8 \\ -7 \\ 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ -4 \\ 2 \end{bmatrix}$ linearly independent?

Use Gaussian elimination to find the echelon form for A.

can't use this as a pivot, so first switch rows...

$$\begin{bmatrix} \boxed{0} & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix} \qquad r_1 \leftrightarrow r_4$$

$$\begin{bmatrix} \boxed{1} & -3 & 2 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 0 & -8 & 5 \end{bmatrix} \qquad \begin{aligned} r_2 &\leftarrow r_2 - 3r_1 \\ r_3 &\leftarrow r_3 + r_1 \end{aligned}$$

rescale .. optional step

$$\begin{bmatrix} \boxed{1} & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & -8 & 5 \end{bmatrix} \qquad \begin{aligned} r_2 &\leftarrow \tfrac{1}{2} r_2 \\ r_3 &\leftarrow \tfrac{1}{2} r_3 \end{aligned}$$

$$\begin{bmatrix} \boxed{1} & -3 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 1 & -1 \\ 0 & -8 & 5 \end{bmatrix} \qquad \begin{aligned} r_3 &\leftarrow r_3 - r_2 \\ r_4 &\leftarrow r_4 + 8r_2 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

can't use this as a pivot

$r_3 \longleftrightarrow r_4$

Echelon form

$$\begin{array}{ccc} P & P & P \end{array}$$
$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot in every column

No free variables ...

Question are the vectors $\begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -7 \\ 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ -4 \\ 2 \end{bmatrix}$ linearly independent? YES

equivalent question does A have any free variables? NO

- This means you can't write any one of those vectors as a linear combination of the other two.
- It means the only solution to $Ax=0$ is $x=0$.

One more Example

$$A = \begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$$

Are the columns linearly independent? No... there are more columns than rows, so there must be free variables.

Find a dependency relationship between the vectors.

Same as solving $Ax=0$ to find an x which is not 0.

Since Right hand side of $Ax=0$ is 0, don't need the augmented matrix.

$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix} \qquad \begin{aligned} r_2 &\leftarrow r_2 + 2r_1 \\ r_3 &\leftarrow r_3 + 4r_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 11 & -5 & 5 \end{bmatrix} \qquad r_3 \leftarrow r_3 - 11r_2$$

Echelon form

$$\begin{array}{cccc} P & P & P & F \end{array}$$
$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 6 & -6 \end{bmatrix} \qquad r_3 \leftarrow \frac{1}{6} r_3$$

Continue to find the reduced echelon form...

$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{aligned} r_2 &\leftarrow r_2 + r_3 \\ r_1 &\leftarrow r_1 + 3r_3 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad r_1 \leftarrow r_1 - 4r_2$$

reduced echelon form

$$\begin{array}{cccc} P & P & P & F \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

didn't change because of the zero's in row 2.

Solve for pivot variables in terms of the free variable

Read of solution to $Ax=0$ ...

$$\begin{cases} x_1 - 3x_4 = 0 \\ x_2 = 0 \\ x_3 - x_4 = 0 \end{cases} \qquad \begin{aligned} x_1 &= 3x_4 \\ x_2 &= 0 \\ x_3 &= x_4 \\ x_4 &= x_4 \end{aligned}$$

In vector parametric form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ 0 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus

$$3\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 0\begin{bmatrix} 4 \\ -7 \\ -5 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = 0$$

a depending relationship is.

$$\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} - 1\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

§1.7 is finished.

Now sections §1.8 and §1.9...

recall

A transformation (or mapping) T is **linear** if

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

and

where f is a linear transformation

$$A = \left[f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \middle| f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \middle| f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right]$$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and so forth...

written in the book as

$$A = \left[T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n) \right]$$

An example of a linear transform (function) that is represented geometrically...

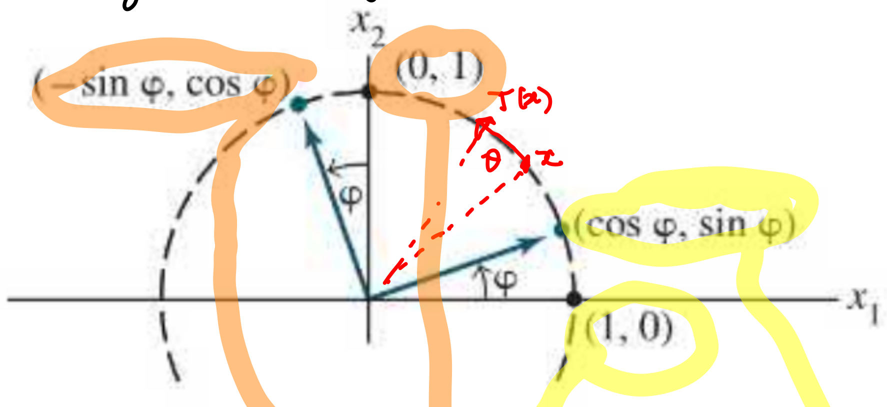

FIGURE 1 A rotation transformation.

What is the matrix corresponding to the above transform?

$T(x)$ means rotate the vector x as shown.

$$A = \left[T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n) \right]$$

$$A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \Bigg| \; T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

This matrix corresponds to rotatation counter-clockwise by φ radians...