From last time...

$$Ax \cdot z = x \cdot A^T z$$

Recall matrix matrix multiplication

$$\underset{m\times n}{A}\ \underset{n\times p}{B} \in \mathbb{R}^{m\times p} \qquad A \in \mathbb{R}^{m\times n} \text{ and } B \in \mathbb{R}^{n\times p}$$

Inner product representation of AB

$$AB = \begin{bmatrix} a_1^T \\ \hline a_2^T \\ \vdots \\ \hline a_m^T \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_p \\ \vdots & & \\ a_m \cdot b_1 & \cdots & a_m \cdot b_p \end{bmatrix}$$

$$\underset{n\times m}{A^T} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \qquad \underset{p\times n}{B^T} = \begin{bmatrix} b_1^T \\ \hline b_2^T \\ \hline \vdots \\ \hline b_p^T \end{bmatrix}$$

Can I multiply these matrices? Yes

$$\underset{p\times n}{B^T}\ \underset{n\times m}{A^T} = \begin{bmatrix} b_1^T \\ \hline b_2^T \\ \hline \vdots \\ \hline b_p^T \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} = \begin{bmatrix} b_1 \cdot a_1 & \cdots & b_1 \cdot a_m \\ \vdots & & \vdots \\ b_p \cdot a_1 & \cdots & b_p \cdot a_m \end{bmatrix}$$

match

$$B^T A^T = \begin{bmatrix} a_1 \cdot b_1 & \cdots & a_m \cdot b_1 \\ \vdots & & \vdots \\ a_1 \cdot b_p & \cdots & a_m \cdot b_p \end{bmatrix} \qquad AB = \begin{bmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_p \\ \vdots & & \\ a_m \cdot b_1 & \cdots & a_m \cdot b_p \end{bmatrix}$$

compare and see one is the transpose of the other

Therefore

$$\boxed{(AB)^T = B^T A^T}$$

and also

$$\boxed{Ax \cdot z = x \cdot A^T z}$$

Example ...

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$$

Compute AB

$$B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 8 & 5 \\ 2 & -4 \end{bmatrix}$$

AB

Compute $B^T A^T$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & 8 & 2 \\ 3 & 5 & -4 \end{bmatrix}$$

$B^T A^T$

Let $A \in \mathbb{R}^{n \times n}$

1. A^T is an invertible matrix.

Why?

- means the same as A is invertible...
- A invertible means $Ax = 0$ has only the trivial solution $x = 0$.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Recall

$$Ax \cdot z = x \cdot A^T z$$

- It's enough to show the only solution to $A^T z = 0$ is $z = 0$.

Suppose there was a $z \neq 0$ such that $A^T z = 0$

Then $x \cdot A^T z = 0$ for any x I like...

Since $x \cdot A^T z = Ax \cdot z$ then $Ax \cdot z = 0$

for every x that I like...

If A is invertible want to show A^T is invertible...

Thus $Ax = z$ can be solved for the z such that $A^T z = 0$.

$Ax = z$ and $Ax \cdot z = 0$

$z \cdot z = 0$

$z_1^2 + z_2^2 + \dots + z_n^2 = 0$

Which means $z \neq 0$.

Therefore the only z such that $A^T z = 0$ is the zero vector $z = 0$. Therefore A^T is invertible.

2.5 LU Factorization:

Matrix-Matrix products..

$A \in \mathbb{R}^{m \times n}$ $\quad B \in \mathbb{R}^{n \times p}$

$f(x) = Ax$ $\quad g(x) = Bx$

$A = \left[f(e_1) \middle| f(e_2) \middle| \dots \middle| f(e_n) \right]$ $\quad B = \left[g(e_1) \middle| g(e_2) \middle| \dots \middle| g(e_p) \right]$

What is the matrix for $f \circ g(x)$. By definition.

$$AB = \left[f \circ g(e_1) \,\middle|\, f \circ g(e_2) \,\middle|\, \cdots \,\middle|\, f \circ g(e_p) \right]$$

Thus

$$\left[f(e_1) \,\middle|\, f(e_2) \,\middle|\, \cdots \,\middle|\, f(e_n) \right] \left[g(e_1) \,\middle|\, g(e_2) \,\middle|\, \cdots \,\middle|\, g(e_p) \right] = \left[f(g(e_1)) \,\middle|\, f \circ g(e_2) \,\middle|\, \cdots \,\middle|\, f \circ g(e_p) \right]$$

$$A \left[b_1 \,\middle|\, b_2 \,\middle|\, \cdots \,\middle|\, b_p \right] = \left[Ab_1 \,\middle|\, Ab_2 \,\middle|\, \cdots \,\middle|\, Ab_p \right]$$

Matrices corresponding to row operations...

Consider elimination step

$$r_i \leftarrow r_i - \alpha r_j \qquad \alpha \in \mathbb{R}, \quad i \neq j$$

invertible. Inverse is

$$r_i \leftarrow r_i + \alpha r_j \qquad \alpha \in \mathbb{R}, \quad i \neq j$$

Note these row operations are linear transformations..

so a matrix corresponds to this operation

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ \vdots \\ x_i - \alpha x_j \\ \vdots \\ x_m \end{bmatrix}$$

the linear function corresponding to $r_i \leftarrow r_i - \alpha r_j$

Denote the matrix by

$$[r_i \leftarrow r_i - \alpha r_j]$$

Therefore

$$f(x) = [r_i \leftarrow r_i - \alpha r_j]\, x \,,$$

What is that matrix?

$$[r_i \leftarrow r_i - \alpha r_j] = \left[\, f(e_1) \,\middle|\, f(e_2) \,\middle|\, \cdots \,\middle|\, f(e_m) \,\right]$$

$$A\left[\, b_1 \,\middle|\, b_2 \,\middle|\, \cdots \,\middle|\, b_p \,\right] = \left[\, Ab_1 \,\middle|\, Ab_2 \,\middle|\, \cdots \,\middle|\, Ab_p \,\right]$$

$$= [r_i \leftarrow r_i - \alpha r_j]\left[\, e_1 \,\middle|\, e_2 \,\middle|\, \cdots \,\middle|\, e_m \,\right]$$

$$[r_i \leftarrow r_i - \alpha r_j] = [r_i \leftarrow r_i - \alpha r_j]\, I$$

Example: $r_2 \leftarrow r_2 - 3r_1$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$r_2 \leftarrow r_2 - 3r_1$

$$\begin{array}{cc} & \text{1st column} \\ \text{2nd row} & \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = [r_2 \leftarrow r_2 - 3r_1]$$

row column

Rule: to find the matrix corresponding to the row operation, just perform the row operation on the identity matrix.

in the 2,1 slot

$$[r_2 \leftarrow r_2 - 3r_1]^{-1} = [r_2 \leftarrow r_2 + 3r_1] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next time use elimination steps to factor matrices...