2.8 Subspaces of $\mathbb{R}^n$

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

a. The zero vector is in H.
b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

These three properties allow you to do vector arithmetic on H.

Two concrete supspace we are interest in...

The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

Let $A \in \mathbb{R}^{m \times n}$, then

$$\text{Col } A = \boxed{\{ Ax : x \in \mathbb{R}^n \}} = \text{range } f \quad \text{where } f(x) = Ax$$

The **null space** of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \boxed{\{ x \in \mathbb{R}^n : Ax = 0 \}} = \text{kernel } f$$

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Goal is to find a basis for the column and null spaces corresponding to a given A.

Example

$$31.\quad A = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline \end{array}$$

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{array}{cccc} P & P & F & F \\ x_1 & x_2 & x_3 & x_4 \end{array} \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row echelon form

It's not the reduced echelon form because the are non-zero entries above one of the pivots

note: this isn't U from the LU factorization because some scaling operations have been performed...

Try the column space...

$$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \}$$

$$= \left\{ \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} x : x \in \mathbb{R}^4 \right\}$$

Want to simplify this characterization of col A by finding a basis for that subspace...

The pivot columns of a matrix A form a basis for the column space of A.

Basis of $\text{Col} \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$ is $\left\{ \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \right\}$

Note: the vectors are linearly independent but still span the column space...

Now try the null space

$$\text{Nul}\,A = \{x \in \mathbb{R}^n : Ax = 0\} = \left\{x \in \mathbb{R}^4 : \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} x = 0 \right\}$$

Solutions of the homogeneous equation

Solve using row operations

$$\begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$$

row operations

Echelon form

$$\sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 - 2r_2$

reduced echelon form

$$\begin{bmatrix} 1 & 0 & -4 & 7 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

convert to algebraic form

$$x_1 - 4x_3 + 7x_4 = 0$$
$$x_2 + 5x_3 - 6x_4 = 0$$

solve for pivot variables

$$x_1 = 4x_3 - 7x_4$$
$$x_2 = -5x_3 + 6x_4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

Solution to $Ax = 0$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 - 7x_4 \\ -5x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \overbrace{x_3 \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}}^{\text{Span}}$$

This part of the vectors ensure they are linearly independent.

Basis for Nul $\begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$ is $\underbrace{\left\{ \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{\# of vectors is the same as the number of free variables in } A.}$.

recall from above...

Basis of Col $\begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$ is $\underbrace{\left\{ \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \right\}}_{\text{\# of vectors is the same as the number of pivot variables in } A.}$

So we've found both Null and Column spaces for the matrix A.

The **dimension** of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.[2]

$\dim \text{Col } A = 2$ $\qquad\qquad$ $\dim \text{Nul } A = 2$

\# of pivots $\qquad\qquad$ \# of free vbls.

Thus, if $A \in \mathbb{R}^{m \times n}$ then $\dim \text{Col } A + \dim \text{Nul } A = n$

Example

$$A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$$

$$\begin{array}{ccccc} P & P & F & P & F \end{array}$$

$$\begin{bmatrix} \boxed{1} & 4 & 8 & 0 & 5 \\ 0 & \boxed{2} & 5 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{echelon form of } A$$

Col A has basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix} \right\}$

$\dim \operatorname{col} A = 3$ also called the rank of A.

Next solve $Ax = 0$. Find reduced echelon form

$$\begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 - 2r_2$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad r_2 \leftarrow \frac{1}{2} r_2$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 7 \\ 0 & 1 & 5/2 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_3 + 7x_5 = 0$$
$$x_2 + \tfrac{5}{2}x_3 - \tfrac{1}{2}x_5 = 0$$
$$x_4 + 4x_5 = 0$$

Solve for the pivots

$$x_1 = 2x_3 - 7x_5$$
$$x_2 = -\tfrac{5}{2}x_3 + \tfrac{1}{2}x_5$$
$$x_4 = -4x_5$$

Solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 7x_5 \\ -\tfrac{5}{2}x_3 + \tfrac{1}{2}x_5 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 1/2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Therefore a basis for Nul A is $\left\{ \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 1/2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$

and $\dim \operatorname{Nul} A = 2$.