Review session for the Exam on Friday Oct 6 will be at 1pm on Oct 5 through zoom at the usual livestream meeting number available on Web Campus.

I'll record it and post a link after it's over.

2.9 Dimension and Rank

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

The following theorem is important for applications and will be needed in Chapters 5 and 6. The theorem (proved in Section 4.5) is certainly plausible, if you think of a p-dimensional subspace as isomorphic to $\mathbb{R}^p$. The Invertible Matrix Theorem

will come back to this in chapter 4.

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$ $\quad$ dim Col A = dim $\mathbb{R}^n = n$

o. rank $A = n$ = dim Col A = # of pivots $\quad$ So there is a pivot in every column and row.

p. dim Nul $A = 0$ = # of free variables $\quad$ Since no free variables then all n are pivots.

q. Nul $A = \{\mathbf{0}\}$

dim $\{0\} = 0$ so no free variables

<u>Another</u> example: Find Col A and Nul A and a matrix N such that Nul A = col N.

$$\textbf{32.}\quad A = \begin{array}{cccc} P & F & P & F \\ \end{array}\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{array}{cccc} P & F & P & F \\ \end{array}\begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So Col A has a basis $\left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\}$. Why?

First write down reduced echelon form:

$$\begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 - \frac{3}{2} r_2$$

$$\begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 9 - \frac{15}{2} = \frac{18-15}{2} = \frac{3}{2}$$

$$r_2 \leftarrow \frac{1}{4} r_2$$

reduced echelon:

$$\begin{array}{cccc} P & F & P & F \\ \end{array}\begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By doing row operations I didn't change the relationships between the columns, but only the rows

$$-3 \overset{P}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \overset{F}{\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}} \qquad -3\, \overset{P}{c_1} = \overset{F}{c_2}$$

I solved for the column corresponding to the free variable...

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$-3\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix}$$

It's true that $-3c_1 = c_2$ for the original matrix.

Find another relationship between the columns...

$$\begin{array}{cccc} P & F & P & F \end{array}$$
$$\begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve for this one

$$\begin{bmatrix} 3/2 \\ 5/4 \\ 0 \end{bmatrix} = 3/2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5/4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad c_4 = \frac{3}{2}c_1 + \frac{5}{4}c_3$$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

Check this

$$\frac{3}{2}(-3) + \frac{5}{4}(-2) = -\frac{9}{2} - \frac{5}{2} = -\frac{14}{2} = -7$$

$$\begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

Check the other components... (also okay?)

Solved for free variables columns in terms of the pivot columns.

$$\text{Col}\,A = \{Ax : x \in \mathbb{R}^4\} = \left\{ \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_i \in \mathbb{R} \right\}$$

$$= \left\{ x_1 \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix} : \quad x_i \in \mathbb{R} \right\}$$

$$-3 \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix} \qquad \begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} + \frac{5}{4} \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

$$= \left\{ \left(x_1 - 3x_2 + \frac{3}{2}x_4\right) \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} + \left(x_3 + \frac{5}{4}x_4\right) \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} : x_i \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\}$$