Some how my note-taking program crashed when the emergency fire message started and I lost the first part of the lecture notes. I'll try to recreate them here as best I remember.

We started by finishing up chapter 2. There was one example that needed to be finished by computing the nullspace and the nullspace matrix N.

32. $$A = \begin{array}{cccc} P & F & P & F \\ \end{array}$$
$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We recalled why the basis for the column space is made from the columns of A corresponding to the pivot variables. The point is that the relationships between the columns of a matrix are unaffected by the row operations and that it's clear the pivot columns in the echelon and reduced echelon forms are independent and that the columns corresponding to the free variables can be written interms of the columns with the pivots.

Thus

$$\text{Col } A = \text{span}\left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\}$$

and $\dim \text{col } A = \text{rank } A = 2 = \#$ of pivot variables

Now we do the nullspace. First

$$\dim \operatorname{Nul} A = \#\text{ of free variables}$$
$$= 4 - \#\text{ of pivot variables} = 2.$$

By definition Nul A is the solutions to the homogeneous problem $Ax=0$.

$$\operatorname{Nul} A = \{x : Ax = 0\}$$

We already know

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \quad \text{is row equivalent to} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ P & F & P & F \end{array}$$
$$\begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form

In algebraic form we have

$$x_1 - 3x_2 + 3/2\, x_4 = 0$$
$$x_3 + 5/4\, x_4 = 0$$

Solving for the pivot variables yields

$$x_1 = 3x_2 - 3/2\, x_4$$
$$x_3 = -5/4\, x_4$$

The solution to $Ax=0$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 3/2\, x_4 \\ x_2 \\ -5/4\, x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}$$

Therefore

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} \right\}.$$

The nullspace matrix is the matrix whose columns are a basis for the nullspace.

$$N = \begin{bmatrix} 3 & -3/2 \\ 1 & 0 \\ 0 & -5/4 \\ 0 & 1 \end{bmatrix}$$

The main point is so we can write

$$\text{Nul } A = \text{Col } N.$$

3.1 Introduction to Determinants

The chapter is very condensed because determinants are less useful for practical computations than for theory.

There is another linear algebra course on theory called Math 430. That course will build more intuition about determinants but also focuses on other things.

Somehow determinants always get short changed.

Consider a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and solving } Ax = b$$

We're not interested right now about the right-hand side of the equation so write the augmented matrix as

$$[A|b] = \left[\begin{array}{cc|c} a & b & ? \\ c & d & ? \end{array}\right]$$

Now, if $a \neq 0$ we could perform the elimination step

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad r_2 = r_2 - \frac{c}{a} r_1$$

$$\begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$

We arrive at an augmented matrix that looks like

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} a & b & ? \\ 0 & \frac{ad-bc}{a} & ? \end{array}\right]$$

which means

$$a x_1 + b x_2 = ?$$

$$\frac{ad-bc}{a} x_2 = ?$$

Solving for x_2 yields

$$x_2 = \frac{a?}{ad-bc}$$

this in the denominator is called the determinant...

We like to call our friends by name and this denominator is so friendly it keeps appearing...

What happens if we swap rows first?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$r_1 \leftrightarrow r_2$

Now eliminate...

$$\begin{bmatrix} \boxed{c} & d \\ a & b \end{bmatrix}$$

$r_2 \leftarrow r_2 - \frac{a}{c} r_1$

$$\begin{bmatrix} c & d \\ 0 & b - \frac{a}{c} d \end{bmatrix} = \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{bmatrix} c & d \\ 0 & \frac{bc-ad}{c} \end{bmatrix}$$

Now solving for x_2 yields

$$x_2 = \frac{c \cdot ?}{bc - ad} = \frac{-c \cdot ?}{ad-bc}$$

here is the determinant again.

One can find determinants for a 3x3 matrix similar. Consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

in this case we rescale the rows ahead of time to prevent fractions, as the algebra is already difficult.

$r_2 \leftarrow a_{11} r_2$

$r_3 \leftarrow a_{11} r_2$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

Elimination

$r_2 \leftarrow r_2 - a_{21} r_1$

$r_3 \leftarrow r_3 - a_{31} r_1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Now do the same thing for the second pivot...

$r_3 \leftarrow (a_{11}a_{22} - a_{12}a_{21}) r_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{33} - a_{13}a_{31}) \end{bmatrix}$$

Now eliminate

$r_3 \leftarrow r_3 - (a_{11}a_{32} - a_{12}a_{31}) r_2$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{33} - a_{13}a_{31}) - (a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}.$$

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Do elimination with 3x3 and discover that

$$x_3 = \frac{?}{a_{11}\Delta}$$

identity in denominator as the determinant

It turns out this expressions called determinants which appear in the denominator when solving Ax=b have a pattern. The pattern can be seen by expressing the determinant of a larger matrix inductively in terms of the determinants of a bunch of smaller matrices.

To define the determinant of a large matrix inductively in terms of smaller matrices, we need a way to create smaller matrices from larger ones.

Given a matrix A let A_{ij} be the submatrix of A formed by deleting the i^{th} row and j^{th} column:

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$A_{2,3} = \begin{bmatrix} 1 & 2 & \cancel{6} \\ \cancel{5} & \cancel{4} & \cancel{1} \\ 1 & 0 & \cancel{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$A_{3,1} = \begin{bmatrix} \cancel{1} & 2 & 6 \\ \cancel{5} & 4 & 1 \\ \cancel{1} & \cancel{0} & \cancel{3} \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

Next time we'll use these submatrices to write down a general expression for an $n\times n$ matrix in terms of determinants of $(n-1)\times(n-1)$ matrices.