(a) Elimination step $\quad r_i \leftarrow r_i - \alpha r_j \qquad \alpha \neq 0, \; i \neq j$

$$\det\left(\left[r_i \leftarrow r_i - \alpha r_j\right] A\right) = \det A$$

(b) row swap $\quad r_i \leftrightarrow r_j \qquad i \neq j$

$$\det\left(\left[r_i \leftrightarrow r_j\right] A\right) = -\det A$$

(c) rescaling $\quad r_i \leftarrow k r_i \qquad$ where $k \neq 0$

$$\det\left(\left[r_i \leftarrow k r_i\right] A\right) = k \det A$$

why is $\det(AB) = (\det A)(\det B)$?

From last time...

Elimination

$$\det\left[r_3 \leftarrow r_3 - 2r_1\right] = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = 1$$

row operation matrix

row swap $\quad r_i \leftrightarrow r_j \qquad i \neq j$

$$\left[r_1 \leftrightarrow r_2\right] = \left[r_1 \leftrightarrow r_2\right]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}, \quad i=1$$

$$= (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13}$$

$$= 0 \det\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 1 \det\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0 \det\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -1 \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1$$

scaling $r_2 \leftarrow 3r_2$

$$[r_2 \leftarrow 3r_2] = [r_2 \leftarrow 3r_2]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1\cdot 3\cdot 1 = 3$$

In summary, the determinants of the matrices corresponding to the row operations are:

$$\begin{cases} \det[r_i \leftarrow r_i - \alpha r_j] = 1 \\ \det[r_i \leftrightarrow r_j] = -1 \\ \det[r_i \leftarrow \alpha r_i] = \alpha \end{cases}$$

(a) Elimination step $r_i \leftarrow r_i - \alpha r_j$ $\quad \alpha \neq 0$, $i \neq j$

$$\det([r_i \leftarrow r_i - \alpha r_j]A) = 1\cdot\det A = \det[r_i \leftarrow r_i - \alpha r_j] \det A$$

(b) row swap $r_i \leftrightarrow r_j$ $\quad i \neq j$

$$\det([r_i \leftrightarrow r_j]A) = -\det A = \det[r_i \leftrightarrow r_j]\det A$$

(c) rescaling $r_i \leftarrow k r_i$ $\quad$ where $k \neq 0$

$$\det([r_i \leftarrow k r_i]A) = k \det A = \det[r_i \leftarrow k r_i]\det A$$

In general $\det EA = \det E \det A$

Recall: One can use row operations to make the reduced row echelon form.

If $A \in \mathbb{R}^{n \times n}$ and invertible then its reduced echelon form is $\boxed{I}$.

To find the inverse of an invertible matrix we mod the augmented matrix

$$[\,A \mid I\,] \xrightarrow{\text{row operations}} [\,I \mid A^{-1}\,]$$

$E_p \cdots E_3 E_2 E_1$

matrices corresponding to the row operations.
elimination + row swaps + rescaling

reduced echelon form of A

Thus

$$E_p \cdots E_3 E_2 E_1 A = I$$

$$\cancel{E_p^{-1}}\, \cancel{E_p} \cdots E_3 E_2 E_1 A = E_p^{-1} I$$

$$E_{p-1} \cdots E_1 \, A = E_p^{-1} I$$

after moving all the matrices over ...

$$A = E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1}$$

Any invertible matrix A is the product of elementary row operations

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Now consider $\det AB$.

$$\det AB = \det E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} B$$

$$= \det E_1^{-1} \left(E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} B \right)$$

$$= \det E_1^{-1} \det \left(E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} B \right)$$

Move the elementary row operation matrices out one at a time

$$= \det E_1^{-1} \det E_2^{-1} \cdots \det E_{p-1}^{-1} \det E_p^{-1} \det B$$

$$= \det E_1^{-1} \det E_2^{-1} \cdots \det E_{p-2}^{-1} \det E_{p-1}^{-1} E_p^{-1} \det B$$

Move the elementary row operation matrices in one at a time

$$= \det E_1^{-1} E_2^{-1} \cdots E_{p-2}^{-1} E_{p-1}^{-1} E_p^{-1} \det B$$

$$= \det A \det B$$

Thus $\det AB = \det A \det B$ $\boxed{\checkmark}$

Now use this property to solve $Ax = b$.

Notation $A_i(b)$ means take the ith column of A and replace it with the vector b.

$A \in \mathbb{R}^{n \times n}$

$$A = \left[a_1 \middle| a_2 \middle| \cdots \middle| a_n \right] \qquad A_1(b) = \left[b \middle| a_2 \middle| \cdots \middle| a_n \right]$$

$$A_2(b) = \left[a_1 \middle| b \middle| \cdots \middle| a_n \right] \qquad A_i(b) = \left[a_1 \middle| \cdots \middle| \underset{\uparrow \text{ ith column}}{b} \middle| \cdots \middle| a_n \right]$$

$$I_1(x) = \left[\, x \;\middle|\; e_2 \;\middle|\; \cdots \;\middle|\; e_n \,\right] \qquad I_2(x) = \left[\, e_1 \;\middle|\; x \;\middle|\; \cdots \;\middle|\; e_n \,\right]$$

What is $\det I_i(x)$?

Answer $\det I_i(x) = x_i$

$$\det I_1(x) = \det \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = x_1$$

$$\det I_2(x) = \det \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} x_2 & 0 \\ 0 & 1 \end{bmatrix} = x_2$$

$$\det I_3(x) = \det \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{bmatrix} = x_3$$

$$I_i(x) = \begin{array}{c} \qquad\qquad i\text{th column} \\ \left[\, e_1 \;\middle|\; \cdots \;\middle|\; x \;\middle|\; \cdots \;\middle|\; e_n \,\right] \end{array}$$

$$A\, I_i(x) = A \left[\, e_1 \;\middle|\; \cdots \;\middle|\; x \;\middle|\; \cdots \;\middle|\; e_n \,\right] = \left[\, Ae_1 \;\middle|\; \cdots \;\middle|\; Ax \;\middle|\; \cdots \;\middle|\; Ae_n \,\right]$$

$$= \left[\, a_1 \;\middle|\; \cdots \;\middle|\; b \;\middle|\; \cdots \;\middle|\; a_n \,\right] = A_i(b)$$

Therefore $\quad A\, I_i(x) = A_i(b)$

$$\det(A\, I_i(x)) = \det(A_i(b))$$

$$\det(A)\underbrace{\det(I_i(x))}_{x_i} = \det(A_i(b))$$

solve for x_i

$$\boxed{x_i = \frac{\det(A_i(b))}{\det(A)}}$$

Cramer's rule. solution to $Ax=b$ in terms of determinants.