In Exercises 3–6, find an explicit description of Nul A by listing vectors that span the null space.

6. $A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$r_1 \leftarrow r_1 - 5r_2$

reduced echelon form

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ P & P & F & F & F \end{array}$$

$$\begin{bmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 6x_3 - 8x_4 + x_5 = 0$

$x_2 - 2x_3 + x_4 = 0$

solve

$x_1 = -6x_3 + 8x_4 - x_5$

$x_2 = 2x_3 - x_4$

Write solution in vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for the nullspace.

Basis for nul(A) is

$$\mathcal{B} = \left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Nullspace matrix

$$N = \begin{bmatrix} -6 & 8 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

column of N are a basis for Nul A

Thus $\text{Nul} A = \text{Col} N$.

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

17. $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix} \in \mathbb{R}^{4\times 2}$ **18.** $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix} \in \mathbb{R}^{4\times 3}$

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 5}$

- Not asking to find Col A or Nul A
- Not even asking what is dim Col A or dim Nul A.

Let $A \in \mathbb{R}^{m\times n}$

$f(x) = Ax$ $f: \mathbb{R}^n \to \mathbb{R}^m$

$\text{Col}\, A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

$\text{Nul}\, A = \{ x : Ax = 0 \} \subseteq \mathbb{R}^n$

17. $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$

$\text{Col}\, A \subseteq \mathbb{R}^4$

$\text{Nul}\, A \subseteq \mathbb{R}^2$

18. $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix} \in \mathbb{R}^{4 \times 3}$

$\text{Col}\, A \subseteq \mathbb{R}^4$

$\text{Nul}\, A \subseteq \mathbb{R}^3$

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{2 \times 5}$

$\text{Col}\, A \subseteq \mathbb{R}^2$

$\text{Nul}\, A \subseteq \mathbb{R}^5$

4.3 Linearly Independent Sets; Bases

An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

also if you make a matrix

$A = [\, v_1 \mid v_2 \mid \cdots \mid v_p \,]$

then the v_j's are dependent if the are free variables..

def of Basis

Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span } \mathcal{B}$$

an independent set of vectors that spans the space

The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in a vector space V, and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

One can replace a vector that's dependent on the others by those others.

(4)

$$= \left\{ x_1\begin{bmatrix}-3\\2\\3\end{bmatrix} + x_2\begin{bmatrix}9\\-6\\-9\end{bmatrix} + x_3\begin{bmatrix}-2\\4\\-2\end{bmatrix} + x_4\begin{bmatrix}-7\\8\\2\end{bmatrix} : x_i \in \mathbb{R} \right\}$$

$$-3\begin{bmatrix}-3\\2\\3\end{bmatrix} = \begin{bmatrix}9\\-6\\-9\end{bmatrix} \qquad \begin{bmatrix}-7\\8\\2\end{bmatrix} = \frac{3}{2}\begin{bmatrix}-3\\2\\3\end{bmatrix} + \frac{5}{4}\begin{bmatrix}-2\\4\\-2\end{bmatrix}$$

$$= \left\{ \left(x_1 - 3x_2 + \tfrac{3}{2}x_4\right)\begin{bmatrix}-3\\2\\3\end{bmatrix} + \left(x_3 + \tfrac{5}{4}x_4\right)\begin{bmatrix}-2\\4\\-2\end{bmatrix} : x_i \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix}-3\\2\\3\end{bmatrix}, \begin{bmatrix}-2\\4\\-2\end{bmatrix} \right\}$$

from lecture 16 example of removing a V_k that was a linear combination of the other vectors and still getting the same span.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

If $S = \text{span}\{v_1, \dots, v_p\}$ then if they are independent they already form a basis. Otherwise use part (a) to remove the dependent vectors one at a time until what's left is a basis...

Note if you start with the span of an infinite number of vectors, you can't remove them one by one to get a basis, because you'd always have an infinite number of vectors left.

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B. ?

recall $\text{row } A = \text{col } A^T$

$A \in \mathbb{R}^{m \times n}$, Suppose $C \in \mathbb{R}^{n \times n}$ is invertible matrix: Then

$$\boxed{\underset{m\times n}{\text{Col } A} = \text{Col } \underset{m\times n \; n\times n}{AC}}$$

Note $A \neq AC$ unless C happened to be I.

$\text{Col } A = \{Ax : x \in \mathbb{R}^n\}$

$\text{col } AC = \{ACz : z \in \mathbb{R}^n\}$

Since C is invertible, then one can solve

$$Cy = x \quad \text{for any } x \in \mathbb{R}^n$$

$$y = C^{-1}x$$

So for every $x \in \mathbb{R}^n$ there is a $y \in \mathbb{R}^n$ such that $Cy = x$

$$\operatorname{Col} A = \{ Ax : x \in \mathbb{R}^n \}$$

$$= \{ Ax : Cy = x \text{ for some } y \in \mathbb{R}^n \}$$

$$= \{ ACy : y \in \mathbb{R}^n \} = \operatorname{col}(AC).$$

In otherwords since $\operatorname{Col} C = \mathbb{R}^n$ then the C mixes up the inputs to A but doesn't change the set of all inputs.

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B. ?

$$A = \underbrace{E_q \cdots E_2 E_1}_{} B$$

row operations.. that is the matrices corresponding to row swaps, elemination and scaling operations...

$$\operatorname{row} A = \operatorname{col}(A^T) = \operatorname{col}\left((E_q \cdots E_2 E_1 B)^T\right)$$

$$= \mathrm{col}\left(B^T \underbrace{E_1^T E_2^T \cdots E_7^T}_{\text{invertible matrices...}} \right) = \mathrm{col}\,(B^T) = \mathrm{row}\, B$$

14. $A = \begin{array}{c} \begin{array}{ccccc} p & F & p & F & p \end{array} \\ \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \end{array}$,

$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find basis for Col A, Nul A and Row A.

New ... Next time

Basis for Col A = $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$

Basis for Nul A = Next time...