If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

$$\begin{array}{ccccc} P & F & P & F & P \end{array}$$

14. $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find:
- Col A
- Nul A
- row A

$$\text{Col } A = \{Ax : x \in \mathbb{R}^5\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$r_2 \leftarrow \frac{1}{5} r_2$

$r_3 \leftarrow -\frac{1}{9} r_3$

reduced echelon form

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 1 & -7/5 & 8/5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 - 5r_3$

$r_2 \leftarrow r_2 - \frac{8}{5} r_3$

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 4x_4 = 0$$
$$x_3 - \tfrac{7}{5}x_4 = 0$$
$$x_5 = 0$$

Solution in vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ \tfrac{7}{5}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Nul}\, A = \{x : Ax = 0\} = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$N = \begin{bmatrix} -2 & -4 \\ 1 & 0 \\ 0 & 7/5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{5\times 2}$$ then, $\text{Nul}\, A = \text{col}\, N = \{Nz : z \in \mathbb{R}^2\}$

Note the Basis for Nul A is $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B. ?

$$\begin{array}{ccccc} P & F & P & F & P \end{array}$$

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$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ -7 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -9 \end{bmatrix} \right\}$$

Note A is also row equivalent to

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It's clear the vectors are independent because of where the 0's are.

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -7/5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Why: For contradiction suppose not. Then

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \qquad c_i \in \mathbb{R}$$

and

$$x = d_1 b_1 + \cdots + d_n b_n \qquad d_i \in \mathbb{R}$$

$b_n \in \mathbb{R}^m$

Then $c_i \neq d_i$ for at least one i.

Subtract $x - x = (c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) - (d_1 b_1 + \cdots + d_n b_n)$

$$0 = (c_1 - d_1) b_1 + \cdots + (c_n - d_n) b_n$$

Since $c_i \neq d_i$ then the coefficient $(c_i - d_i) \neq 0$ and so this is a dependency relation between the b's.

By assumption the b_i's are independent, so that couldn't have happened... this is a contradiction so the theorem is true.

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Another approach... $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$.

Then $V = \text{Col } B$ because the basis spans.

Solving $B\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = x$ for c gives one unique solution.

Why? This happens when B has no free variables... which means there is a pivot in every column...

Since the columns of B are independent, there is a pivot in every column... otherwise they wouldn't be independent...

Note in section 4.5 we prove

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

It says that if you know the dimension of a space and you have the right number of vectors, then

independence implies spanning

and/or,

spanning implies independence

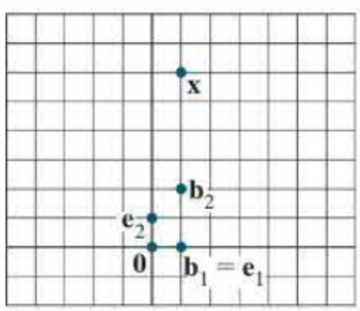

FIGURE 1 Standard graph paper.

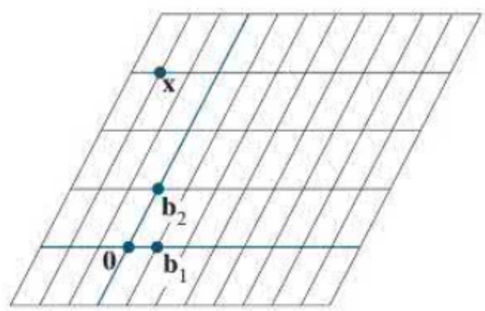

FIGURE 2 $\mathcal{B}$-graph paper.

Basis elements may not be perpendicular to each other, because independence only means not the same (when there are two vectors)

Inconvenient basis is not orthogonal (cartesian). We'll fix this with column operations and the Gram-Schmidt algorithm in the future...