$\mathcal{B} = \{w_1, w_2, \ldots, w_p\}$ assume these are a basis

$\mathcal{C} = \{v_1, v_2, \ldots, v_p\}$

so can write the v's in terms of this basis.

$$\begin{cases} v_1 = w_1 c_{11} + w_2 c_{21} + \cdots + w_p c_{p1} \\ v_2 = w_1 c_{12} + w_2 c_{22} + \cdots + w_p c_{p2} \\ \vdots \\ v_p = w_1 c_{1p} + w_2 c_{2p} + \cdots + w_p c_{pp} \end{cases}$$

$$[v_1]_{\mathcal{B}} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{p1} \end{bmatrix} \quad \cdots \quad [v_p]_{\mathcal{B}} = \begin{bmatrix} c_{1p} \\ c_{2p} \\ \vdots \\ c_{pp} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & & \ddots & \\ c_{p1} & c_{p2} & \cdots & c_{pp} \end{bmatrix}$$

then $A = BC$ where $A = [v_1 | \cdots | v_p]$, $B = [w_1 | \cdots | w_p]$

$[v_i]_{\mathcal{B}}$ can be found for every vector...

Those are from $\mathcal{C}$

$$C = \left[[v_1]_{\mathcal{B}} \,\middle|\, [v_2]_{\mathcal{B}} \,\middle|\, \cdots \,\middle|\, [v_p]_{\mathcal{B}} \right]$$

rewriting stuff from $\mathcal{C}$ in the $\mathcal{B}$ basis

$\mathcal{B} = \{w_1, w_2, \ldots, w_p\}$ $\quad$ $\mathcal{C} = \{v_1, v_2, \ldots, v_p\}$

Let $x \in V$. Then $[x]_{\mathcal{B}} = c$ $\quad$ $A = [v_1 | \cdots | v_p]$

$x = w_1 c_1 + w_2 c_2 + \cdots + w_p c_p = Ac$ $\quad$ $B = [w_1 | \cdots | w_p]$

columns of B are linearly independent... no free variables, so the linear function represented by B is one-to-one

Also $[x]_{\mathcal{C}} = d$

$x = v_1 d_1 + v_2 d_2 + \cdots + v_p d_p = Bd$

Recall that $\boxed{A = BC}$

Since $x = Ac$ and $x = Bd$ then $Ac = Bd$

$BCc = Bd$

since B is one-to-one

$Cc = d$

Therefore $C\,[x]_{\mathcal{B}} = [x]_{\mathcal{C}}$

this is called the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$

$C = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [\mathcal{C} \leftarrow \mathcal{B}]$

book notation

my notation today...

Change of Basis Theorem

$[\mathcal{C} \leftarrow \mathcal{B}]\,[x]_{\mathcal{B}} = [x]_{\mathcal{C}}$

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

Examples ...

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.
   a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
   b. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).

(a) $$[\mathcal{C} \leftarrow \mathcal{B}] = \begin{bmatrix} [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} \end{bmatrix}$$

$$[b_1]_{\mathcal{C}} = [6c_1 - 2c_2]_{\mathcal{C}} = 6[c_1]_{\mathcal{C}} - 2[c_2]_{\mathcal{C}} = 6\begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}6\\-2\end{bmatrix}$$

$$c_1 = 1c_1 + 0c_2 = [c_1 | c_2]\begin{bmatrix}1\\0\end{bmatrix} \qquad [c_1]_{\mathcal{C}} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$c_2 = 0c_1 + 1c_2 = [c_1 | c_2]\begin{bmatrix}0\\1\end{bmatrix} \qquad [c_2]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$[b_2]_{\mathcal{C}} = [9\mathbf{c}_1 - 4\mathbf{c}_2] = \begin{bmatrix}9\\-4\end{bmatrix}$$

$$[\mathcal{C} \leftarrow \mathcal{B}] = \begin{bmatrix} [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix}6 & 9\\-2 & -4\end{bmatrix}$$

b. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).

$$[x]_{\mathcal{C}} = [\mathcal{C} \leftarrow \mathcal{B}][x]_{\mathcal{B}} = [\mathcal{C} \leftarrow \mathcal{B}][-3b_1 + 2b_2]_{\mathcal{B}}$$

$$[-3b_1 + 2b_2]_{\mathcal{B}} = -3[b_1]_{\mathcal{B}} + 2[b_2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$[x]_{\mathcal{C}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} ? \\ \vdots \end{bmatrix}$$

For each subspace in Exercises 1–8, (a) find a basis, and (b) state the dimension.

5. $\left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

$$\left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} \right. = a \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Therefore

$$\left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\} = \operatorname{Col} \begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix}$$

Find the echelon form

$$\begin{array}{c} \begin{array}{ccc} P & P & F \end{array} \\ \begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix} \end{array}$$

$r_2 \leftarrow r_2 - 2r_1$
$r_3 \leftarrow r_3 + r_1$
$r_4 \leftarrow r_4 + 3r_1$

basis

$$\begin{bmatrix} 1 & -4 & -2 \\ 0 & 13 & 0 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{bmatrix}$$

$r_3 \leftarrow r_3 + \frac{4}{13} r_2$
$r_4 \leftarrow r_4 + \frac{5}{13} r_2$

$$\begin{array}{c} \begin{array}{ccc} P & P & F \end{array} \\ \begin{bmatrix} 1 & -4 & -2 \\ 0 & 13 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

(a) Therefore

$$\left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\} = \mathrm{Col} \begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix} = \operatorname{span} \mathcal{B}$$

where $\mathcal{B}$ is the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \right\}$$

(b) $$\dim \left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\} = 2$$