Eigenvector ~ Eigenvalue Problem

Let $A \in \mathbb{R}^{n\times n}$

Solve $Ax = \lambda x$ for $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

where $x \neq 0$ but $\lambda = 0$ is allowed

Why?

$$\underbrace{Ax}_{\text{Matrix-vector multiplication}} = \underbrace{\lambda x}_{\text{Scalar-vector multiplication}}$$

The idea is to convert Matrix-vector multiplication into scalar-vector multiplication, which is much simpler...

Invertible and non-invertible matrices in $\mathbb{R}^{n\times n}$.

either not square
or not enough pivots...

not enough pivots means there are free variables...
If there are free variables then $Ax=0$ has lots of solutions... the set of solutions is

$$\text{Nul}\,A = \{x : Ax = 0\}.$$

If $\lambda = 0$ then $Ax = \lambda x$ reduces to $Ax=0$.

Result. If $A \in \mathbb{R}^{n\times n}$ and A is non-invertible then $\lambda = 0$ is an eigenvalue and the eigenvectors are any $x \in \text{Nul}\,A$ such that $x \neq 0$.

Note also, if $A \in \mathbb{R}^{n \times n}$ then $\det A = 0$ exactly when A is non-invertible.

(If $\det A \neq 0$ then A is invertible).

Idea Use determinants to find eigenvalues...

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0$$

Trying to find x's such that $x \neq 0$ and they satisfy the above equation

That means $x \in \text{Nul}(A - \lambda I)$ and $x \neq 0$.

We need to find values of λ such that $A - \lambda I$ has a large null space...

In otherwords $A - \lambda I$ has free variables

In otherwords $A - \lambda I$ is not invertible

In otherwords $\det(A - \lambda I) = 0$.

Example...

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{so,} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

characteristic polynomial of A.

$$\chi_A(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \underbrace{(1-\lambda)(4-\lambda)(6-\lambda)}_{\text{polynomial in } \lambda \text{ of degree } 3.}$$

In general if $A \in \mathbb{R}^{n\times n}$ then $\det(A-\lambda I)$ is a polynomial of degree n in λ,

Recall the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ are just the entries on the diagonal, that is, $\lambda = 1, 4$ and 6.

These are also the solution to the polynomial equation

$$\chi_A(\lambda) = \det(A-\lambda I) = 0.$$

<u>Method</u>: for solving $Ax = \lambda x$ for x and λ with $x \neq 0$.

1. find $\chi_A(\lambda) = \det(A-\lambda I)$.
2. Solve for the roots $\chi_A(\lambda) = 0$.
3. Let $x \in \mathrm{Nul}(A-\lambda I)$ where $x \neq 0$ and λ are the roots found above

Now the pairs λ, x are solutions to $Ax = \lambda x$.

Note make sure to indicate which λ's go with which x's when writing down a solution

Similarity between Matrices

A is similar to B

means

there exists an invertible matrix P such that $A = PBP^{-1}$.

Transitivity: If A is similar to B

and B is similar to C

Then A is similar to C.

Why?

By hypothesis there is P such that $A = PBP^{-1}$

and Q such that $B = QCQ^{-1}$

Need to find a matrix R such that $A = RCR^{-1}$.

Guess: $R = PQ$ plug it in and check

$$RCR^{-1} = PQ\,C\,\boxed{(PQ)^{-1}} = PQ\,C\,Q^{-1}P^{-1}$$

$$= P(QCQ^{-1})P^{-1} = PBP^{-1} = A$$

What is the point of a similarity?

Suppose A is similar to B then there is P such that $A = PBP^{-1}$.

What is χ_A and how does it compare with χ_B?

$$\chi_A(\lambda) = \det(A - \lambda I) = \det(\underbrace{PBP^{-1} - \lambda I}_{})$$

I know how to simplify determinants of product but not differences.

Idea ... make factors ...

$$PBP^{-1} - \lambda I = PBP^{-1} - \lambda PP^{-1} = (PB - \lambda P)P^{-1}$$

since λ is a scalar

$$= (PB - P\lambda)P^{-1} = P(B - I\lambda)P^{-1}$$

$$\chi_A(\lambda) = \det P(B - I\lambda)P^{-1} = \cancel{\det P} \det(B - I\lambda) \cancel{\det(P^{-1})}$$

since $\det P \det P^{-1} = \det(PP^{-1}) = \det I = 1$

$$\chi_A(\lambda) = \det(B - I\lambda) = \chi_B(\lambda).$$

So A similar to B means $\chi_A(\lambda) = \chi_B(\lambda)$.