$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

find the eigenvalues and eigenvectors of A

① Find the characteristic polynomial.

characteristic polynomial of A.

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \underbrace{(1-\lambda)(4-\lambda)(6-\lambda)}_{\text{polynomial in } \lambda \text{ of degree 3.}}$$

② Find the roots of the polynomial

$$\chi_A(\lambda) = 0 \qquad (1-\lambda)(4-\lambda)(6-\lambda) = 0$$

$$\lambda = \underbrace{1, 4, 6}_{\text{eigenvalues.}}$$

③ Find non-zero vectors $x \in \mathrm{Null}(A - \lambda I)$ where λ is an eigenvalue

$\lambda = 1$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \qquad r_2 \leftrightarrow r_3$$

Pivots

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 3 & 5 \end{bmatrix}$$

since we know there are only two pivots, there must be row operations that make the last row zero... no need to write them down..

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_2 \leftarrow \frac{1}{5} r_2$$

$$\begin{bmatrix} 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 - 3r_2$$

$$\begin{bmatrix} 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow \frac{1}{2} r_1$$

reduced echelon form

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$\lambda = 1$

Solutions to $(A - \lambda I)x = 0$ are

$$x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

eigen vector.

Thus $\lambda = 1$ $\quad x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvalue-eigenvector pair for A

$\lambda = 4$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \begin{bmatrix} \boxed{-3} & 2 & 3 \\ 0 & 0 & \boxed{5} \\ 0 & 0 & 2 \end{bmatrix}$$

already have 2 pivots so know already the last rows will be zero.

$$\begin{bmatrix} \boxed{-3} & 2 & 3 \\ 0 & 0 & \boxed{5} \\ 0 & 0 & 0 \end{bmatrix} \qquad r_2 \leftarrow \frac{1}{5} r_2$$

$$\begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 - 3r_2$$

$$\begin{bmatrix} -3 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow -\tfrac{1}{3} r_1$$

reduced Echelon form

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - \tfrac{2}{3}x_2 &= 0 \\ x_2 &= x_2 \leftarrow \text{free variable} \\ x_3 &= 0 \end{aligned}$$

Solutions $\quad x = \begin{bmatrix} 2/3\, x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}$

eigenvector

Thus $\lambda = 4$ and $x = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is another eigenvalue eigenvector pair.

note I set $x_2 = 3$ to avoid fractions...

Note, the eigenvectors will be used as a basis later and unit vector make for a convenient basis.

Note, could also rescale the eigenvector to be a unit vector...

$\lambda = 6$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 + r_2$$

$$\begin{bmatrix} \boxed{-5} & 0 & 8 \\ 0 & \boxed{-2} & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow -\frac{1}{5} r_1$

$r_2 \leftarrow -\frac{1}{2} r_2$

reduced echelon form

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} \boxed{1} & 0 & -8/5 \\ 0 & \boxed{1} & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \frac{8}{5} x_3 = 0$$
$$x_2 - \frac{5}{2} x_3 = 0$$

Solution

$$x = \begin{bmatrix} 8/5\, x_3 \\ 5/2\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}$$

eigenvector.

Thus $\lambda = 6$ and $x = \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}$ is another eigenvalue eigenvector pair.

In summary: Complete solution to the eigenvalue–eigenvector problem...

λ	x
1	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
4	$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$
6	$\begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}$

Idea consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix} \right\}$$

Let $P = \begin{bmatrix} 1 & 2 & 8/5 \\ 0 & 3 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}$

$$AP = A \begin{bmatrix} 1 & 2 & 8/5 \\ 0 & 3 & 5/2 \\ 0 & 0 & 1 \end{bmatrix} = \left[A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; A\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \;\middle|\; A\begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix} \right]$$

eigenvectors ... so replace A by the corresponding eigenvalue ...

$$= \left[1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; 4\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \;\middle|\; 6\begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 2 & 8/5 \\ 0 & 3 & 5/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

eigenvalues of A on the diagonal...

Therefore

$$AP = PD$$

Note since the columns of P are a basis, then P is invertible...

Then $$A\cancel{P}\,\cancel{P^{-1}} = PDP^{-1}$$

$$A = PDP^{-1}$$

Conclusion is that A is similar to D.

recall

Similarity between Matrices

A is similar to B

means

there exists an invertible matrix P such that $A = PBP^{-1}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \text{ is similar to } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

In general, if $A \in \mathbb{R}^{n \times n}$ is a matrix and $\chi_A(\lambda)$ has n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$

Then the corresponding eigenvectors of A form a basis.

$$\beta = \{x_1, x_2, \ldots, x_n\}.$$

Note that $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2, \ldots$, $Ax_n = \lambda_n x_n$

Then defining $P = \left[x_1 \,\middle|\, x_2 \,\middle|\, \cdots \,\middle|\, x_n \right]$ yields

$$A = PDP^{-1} \quad \text{where } D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}.$$

already know this

$$\det A = \det D$$

$$\det(A - \lambda I) = \det(D - \lambda I)$$

$$\chi_A(\lambda) = \chi_D(\lambda)$$

More...

$$A^2 = AA = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$$

$$= P \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 4^2 & 0 \\ 0 & 0 & 6^2 \end{bmatrix} P^{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}^2 = P \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 4^2 & 0 \\ 0 & 0 & 6^2 \end{bmatrix} P^{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}^3 = P \begin{bmatrix} 1^3 & 0 & 0 \\ 0 & 4^3 & 0 \\ 0 & 0 & 6^3 \end{bmatrix} P^{-1}$$

If one can take powers so easily, it's also clear what the roots are

$$\sqrt{\begin{bmatrix} 1 & 2 & 3 \\ 0 & .4 & 5 \\ 0 & 0 & 6 \end{bmatrix}} = P \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix} P^{-1}$$

Check

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & .4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = P \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix} P^{-1} P \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix} P^{-1}$$