9. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

Find eigenvalues and eigenvectors...

① Find the λ's using $\chi_A(\lambda) = 0$

② Find the x's using $\mathrm{Nul}(A - \lambda I)$.

①
$$\chi_A(\lambda) = \det\begin{pmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{pmatrix} = (3-\lambda)(5-\lambda) + 1$$

$$= \lambda^2 - 8\lambda + 15 + 1 = \lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)(\lambda - 4) = 0$$

$\lambda = 4$ of multiplicity 2...

②
$$\mathrm{Nul}(A - 4I) = \mathrm{Nul}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

Solve
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} x = 0$$

$r_2 \leftarrow r_2 + r_1$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow -r_1$

$$\begin{array}{cc} P & F \end{array}$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$x_1 = -x_2 \leftarrow$ free variable

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\leftarrow$ eigenvector...

In $\mathbb{R}^2$ we have $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\chi_A(\lambda) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

Eigenvalues ... use quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad a=1, \quad b=-(a_{11}+a_{22})$$
$$c = a_{11}a_{22} - a_{12}a_{21}$$

If

$$b^2 - 4ac = (a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$$

$$= \underbrace{a_{11}^2 + 2a_{11}a_{22} + a_{22}^2 - 4a_{11}a_{12}}_{\text{factor this}} + 4a_{12}a_{21}$$

$$= a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{12}a_{21}$$

$$= \underbrace{(a_{11}-a_{22})^2}_{\text{positive}} + \underbrace{4a_{12}a_{21}}_{\text{what's the sign of this...}} \geq 0$$

then the eigenvalues are real.

Note. if $A = A^T$ the matrix is called symmetric and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

thus $A = A^T$ implies $a_{12} = a_{21}$

In this case $b^2 - 4ac = (a_{11}-a_{22})^2 + 4a_{12}^2 \geq 0$

Conclusion if $A \in \mathbb{R}^{2\times 2}$ and $A = A^T$ then the eigenvalues are real.

Spectral Theorem for Symmetric Matrices: If $A \in \mathbb{R}^{n\times n}$ and $A = A^T$ then the eigenvalues are real and the eigenvector can be taken to form a orthonormal basis.

Before this, we go back to complex eigenvalues

Complex: Idea define i to be the square root of -1.

means that $i^2 = -1$.

Note that if i is a square root of -1 then also $-i$ is a square root of -1.

Multiplication of 2 complex numbers. (foil method)

$$(1+2i)(2-3i) = 2 - 6i^2 - 3i + 4i = 8 + i$$

Conjugation

$$\overline{1+2i} = 1-2i$$

swaps $-i$ for i

both are square roots of -1.

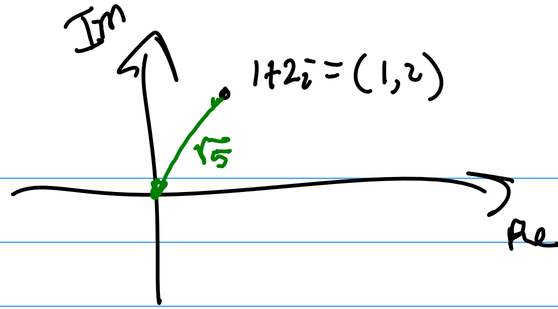
$$\overline{(1+2i)(2-3i)} = \overline{8+i} = 8-i$$

$$\overline{(1+2i)}\,\overline{(2-3i)} = (1-2i)(2+3i) = 8-i$$

Absolute value (modulus or norm)

$$|1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$$

the distance to the origin



Vectors

$$\overline{\begin{bmatrix} 1+i \\ 3-i \end{bmatrix}} = \begin{bmatrix} 1-i \\ 3+i \end{bmatrix}$$

Matrices

$$\overline{\begin{bmatrix} 2+3i & 1-5i \\ 2+i & 1-3i \end{bmatrix}} = \begin{bmatrix} 2-3i & 1+5i \\ 2-i & 1+3i \end{bmatrix}$$

why... so that

$$\overline{Ax} = \bar{A}\bar{x}$$

works...

Case when eigenvalues are complex.

$A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{C}$ and $\underbrace{\lambda \neq \bar{\lambda}}_{\lambda \text{ is truly complex}}$.

Let x be the eigenvector corresponding to λ. Then

$$Ax = \lambda x$$

$$\overline{Ax} = \overline{\lambda x}$$

$$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

$$\boxed{A\bar{x} = \bar{\lambda}\bar{x}}$$

This tells me $\bar{\lambda}$ is another eigenvalue with eigenvector $\bar{x}$.

Q: Could it happen that $\underbrace{\lambda \neq \bar{\lambda}}_{\lambda \text{ is complex}}$ but that $x \in \mathbb{R}^n$

No. Because if $x \in \mathbb{R}^n$ then $Ax \in \mathbb{R}^n$ but $\bar{\lambda}x \notin \mathbb{R}^n$

Thus $x \neq \bar{x}$. That is x is truly complex.

Summary: If $\lambda \neq \bar{\lambda}$ is complex then $x \neq \bar{x}$ is complex.

Could a real eigenvalue have a corresponding eigenvector x which is truly complex... i.e. $\lambda = \bar{\lambda}$ but $x \neq \bar{x}$.

Suppose $Ax = \lambda x$ and both $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

then define $y = \alpha x$ where $\alpha \in \mathbb{C}$ with $\alpha \neq \bar{\alpha}$.