Let $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$

Suppose $Ax = \lambda x$ and $\lambda \notin \mathbb{R}$ and $x \notin \mathbb{R}^n$

also know $A\bar{x} = \bar{\lambda}\bar{x}$ so $\bar{\lambda}$ and $\bar{x}$ also form an eigenvalue-eigenvector pair

$\lambda = \alpha + i\beta$, $\quad \alpha, \beta \in \mathbb{R}$

$x = u + iv \quad u, v \in \mathbb{R}^n$

$$Ax = A(u+iv) = Au + Aiv = Au + iAv$$

i is a scalar

$$\|$$

$$\lambda x = (\alpha + i\beta)(u + iv) = \alpha u - \beta v + i\alpha v + i\beta u$$

$$= (\alpha u - \beta v) + i(\alpha v + \beta u)$$

Thus, $Au = \alpha u - \beta v$ and $Av = \alpha v + \beta u$

combine this

$$A\left[u \middle| v\right] = \left[Au \middle| Av\right] = \left[\alpha u - \beta v \middle| \alpha v + \beta u\right]$$

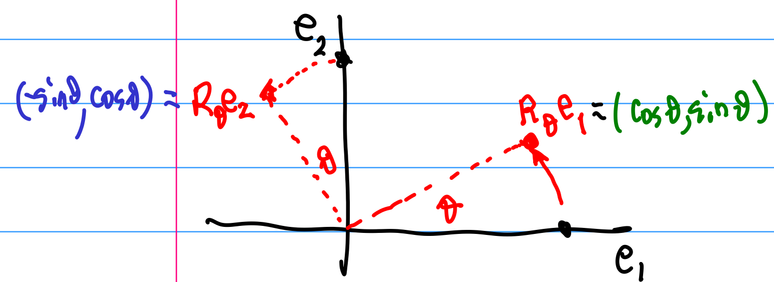
$$= \left[u \middle| v\right]\begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix}$$

Therefore $\quad A\begin{bmatrix} u \mid v \end{bmatrix} = \begin{bmatrix} u \mid v \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

$$A\begin{bmatrix} \mathrm{Re}\,x \mid \mathrm{Im}\,x \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\,x \mid \mathrm{Im}\,x \end{bmatrix}\begin{bmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{bmatrix}$$

This matrix has lots of structure $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$.

What is the matrix for a rotation by θ in 2D ?



$$M = \begin{bmatrix} R_\theta e_1 \mid R_\theta e_2 \end{bmatrix}$$

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Polar coordinates:

$$\alpha = \rho\cos\theta \qquad \alpha^2+\beta^2 = \rho^2(\cos^2\theta+\sin^2\theta) = \rho^2$$

$$-\beta = \rho\sin\theta \qquad \rho = \sqrt{\alpha^2+\beta^2} = |\lambda|$$

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} \rho\cos\theta & -\rho\sin\theta \\ \rho\sin\theta & \rho\cos\theta \end{bmatrix} = \rho\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \rho\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Therefore

$$A\begin{bmatrix} u \mid v \end{bmatrix} = \begin{bmatrix} u \mid v \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} u \mid v \end{bmatrix}\begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A\begin{bmatrix} u \mid v \end{bmatrix} = \begin{bmatrix} u \mid v \end{bmatrix}\begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Point: Complex eigenvalues correspond to rotations between the corresponding eigenvectors along with a scalar multiplication of $|\lambda|$.

Remark: There are no more complex numbers in the above interpretation.

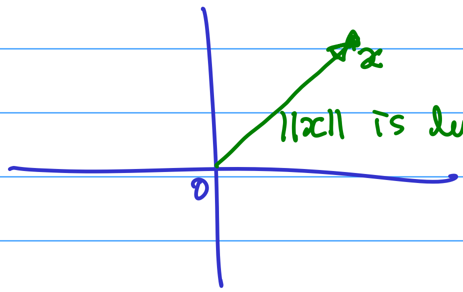
6.1 Inner Product, Length, and Orthogonality

$x \in \mathbb{R}^n$ then $\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} = \sqrt{x \cdot x}$

length of the vector x or norm

since the x's are real.

Generalization of the Pythagorean theorem...



$\|x\|$ is length of vector measured from the origin...

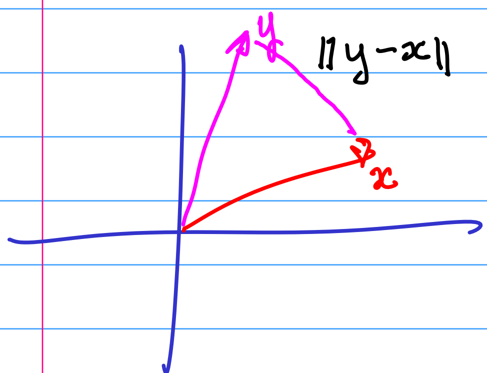
Dot product:

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ $\qquad x \cdot y = x^T y = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$

Recall that a linear function $f(x) = Ax$ naturally consists of coefficients multiplying the elements of x. Those are the dot products we saw before...

If two vectors are perpendicular, there dot product is zero....

$x, y \in \mathbb{R}^n$



$\|y-x\|$ is the distance between the ends of the x and y vectors..

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

since x's are real

$$= \sqrt{x \cdot x}$$

Therefore:

$$\|y-x\| = \sqrt{(y-x)\cdot(y-x)}$$

or

$$\|y-x\|^2 = (y-x)\cdot(y-x) = y\cdot y + x\cdot x \underbrace{- y\cdot x - x\cdot y}$$

order does change the dot prod.

$$= \|y\|^2 + \|x\|^2 - 2y\cdot x$$

If $x \perp y$ then Pythagorean says $\|y-x\|^2 = \|y\|^2 + \|x\|^2$

which means $y\cdot x = 0$.