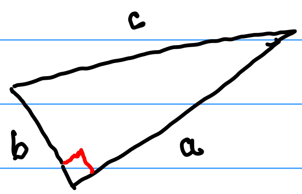
Pythagorean theorem

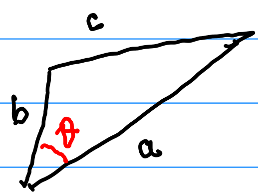


$a^2 + b^2 = c^2$

if and only if

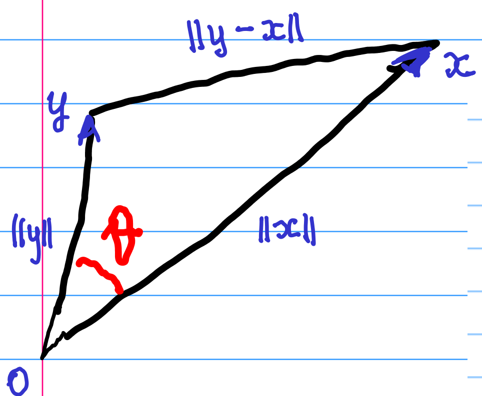
the angle between a and b is a right angle...

Law of cosines



$a^2 + b^2 = c^2 + 2ab\cos\theta$

$\|x\|^2 + \|y\|^2 = \|y-x\|^2 + 2\|x\|\,\|y\|\cos\theta$



compare

from last time

$\|y-x\| = \sqrt{(y-x)\cdot(y-x)}$

or

$\|y-x\|^2 = (y-x)\cdot(y-x) = y\cdot y + x\cdot x - y\cdot x - x\cdot y$

order does change the dot prod.

$= \|y\|^2 + \|x\|^2 - 2y\cdot x$

Thus, $\quad 2y\cdot x = 2\|x\|\,\|y\|\cos\theta$

$y\cdot x = \|x\|\,\|y\|\cos\theta$

here θ is the angle between the vectors x and y.

Since dot products are used for matrix vector multiplication we can use linear algebra to do geometry.

and similarly geometry to solve linear algebra problems.

solve for the angle (or at least $\cos\theta$)

$$x \cdot y = \|x\|\|y\| \cos\theta$$

$$\cos\theta = \frac{x \cdot y}{\|x\|\,\|y\|} = \frac{x}{\|x\|} \cdot \frac{y}{\|y\|} = u \cdot v$$

where

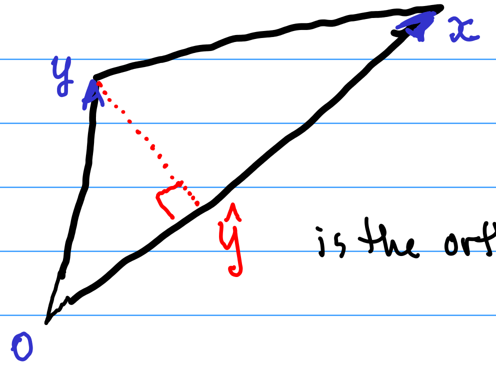
$u = \frac{x}{\|x\|}$ unit vector in the x direction

$v = \frac{y}{\|y\|}$ unit vector in the y direction

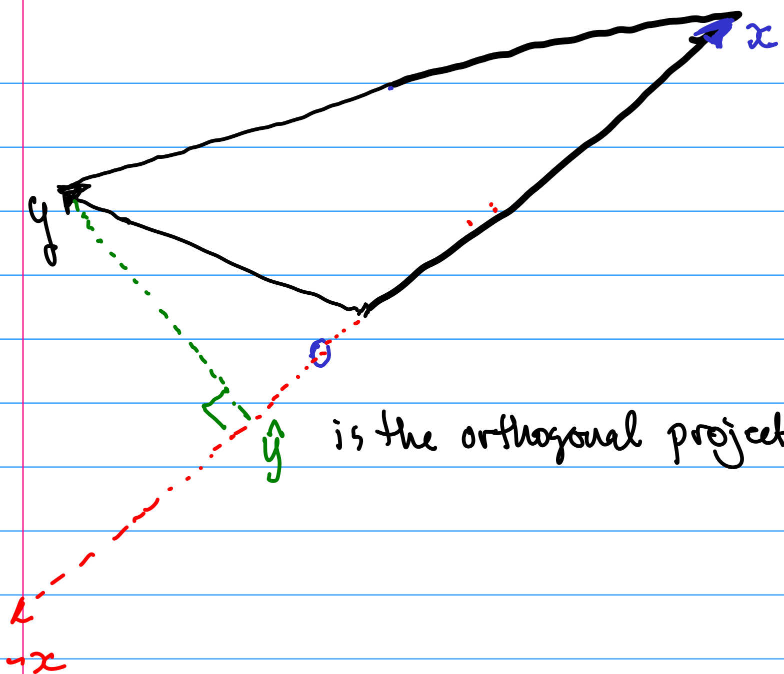
Note: in other courses the unit vector in the x direction is often denoted by $\hat{x}$.

in the next section our book uses $\hat{x}$ to mean something else, so we will follow the book...

<u>Orthogonal Projections</u>

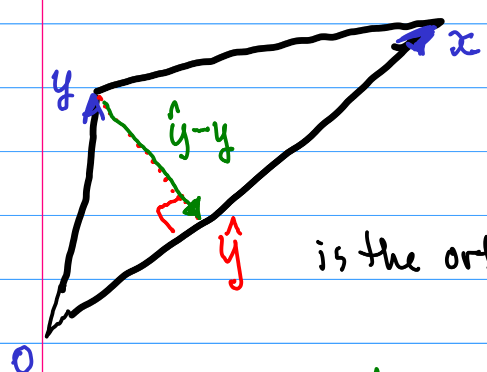


is the orthogonal projection of y on the vector x.



$\hat{y}$ is the orthogonal projection of y on the vector x.

For simplicity we'll work with an accute angle...



$\hat{y}$ is the orthogonal projection of y on the vector x.

$\hat{y}-y$ needs to be perpendicular to x

Thus $(\hat{y}-y)\cdot x = 0$ Now solve for $\hat{y}$.

Since $\hat{y}$ lies in the direction (or opposite direction) of the x vector then $\hat{y} = \alpha x$ for some $\alpha \in \mathbb{R}$.

Substitute

$$(\alpha x - y)\cdot x = 0$$

$$\alpha x \cdot x = y \cdot x$$

So

$$\alpha = \frac{y\cdot x}{x\cdot x} = \frac{y\cdot x}{\|x\|^2} = \frac{x\cdot y}{\|x\|^2}$$

want a function f such that

unit vector of x

$$f(y) = \hat{y} = \alpha x = \left(\frac{x\cdot y}{\|x\|^2}\right)x = \left(\frac{x}{\|x\|}\cdot y\right)\frac{x}{\|x\|} = (u\cdot y)u$$

$$= u(u\cdot y) = uu^Ty = (uu^T)y$$

$$= Py$$

recall

$$u = \frac{x}{\|x\|}$$

where $P = uu^T$ is a matrix.

Remark, the mapping $y \to \hat{y}$ is linear. That is, the orthogonal projection of y onto the vector x can be represented by matrix multiplication

Orthogonal complements

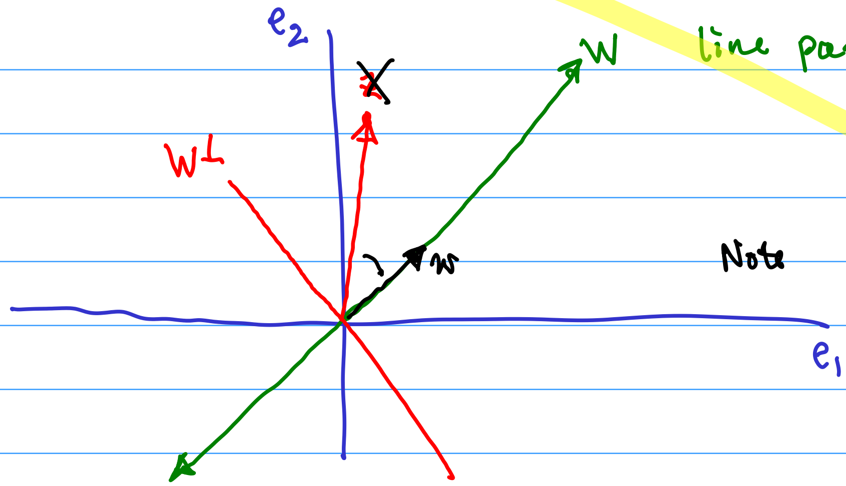
Orthogonal Complements

To provide practice using inner products, we introduce a concept here that will be of use in Section 6.3 and elsewhere in the chapter. If a vector **z** is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then **z** is said to be **orthogonal to** W. The set of all vectors **z** that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (and read as "W perpendicular" or simply "W perp").

Let $W \subseteq \mathbb{R}^n$ and W is a subspace.

$$W^\perp = \{ z : z\cdot w = 0 \text{ for all } w \in W \}$$

Example in $\mathbb{R}^2$



line passing through the origin

$\dim W = 1$

$\dim W^\perp = 1$

Note $\dim W + \dim W^\perp = 2$

In $\mathbb{R}^n$ we also have $\dim W + \dim W^\perp = n$

in 6.3 we note that any vector $z \in \mathbb{R}^n$ can be represented as the sum of a vector in W and a vector in $W^\perp$.

6.2 Orthogonal Sets

After Thanksgiving...