6.2 Orthogonal Sets

After Thanksgiving...

- Let $\{u_1, u_2, \ldots, u_p\} \subseteq \mathbb{R}^n$ then this is called an **orthogonal set** if $u_i \cdot u_j = 0$ whenever $i \neq j$.

- Let $\{u_1, u_2, \ldots, u_p\} \subseteq \mathbb{R}^n$ then this is called an **orthonormal set** if $u_i \cdot u_j = \begin{cases} 0 & \text{whenever } i \neq j. \\ 1 & \text{whenever } i = j. \end{cases}$

- Orthogonal basis is a basis which is also an orthogonal set

- Orthonormal basis is a basis which is also an orthonormal set

14 If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Why? Suppose for contradiction that $u_1, \ldots, u_p$ are orthogonal but not independent.

We need to show this can't happen. That means the vectors must be independent.

If the u's are dependent then there exists c_i's not all zero such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_p u_p = 0 .$$

Since the c_i's are not all zero then there is some i for which $c_i \neq 0$. Take dot product with u_i

$$u_i \cdot (c_1 u_1 + c_2 u_2 + \cdots + c_p u_p) = u_i \cdot 0$$

$$c_1 u_i \cdot u_1 + c_2 u_i \cdot u_2 + \cdots + c_i u_i \cdot u_i + \cdots + c_p u_i \cdot u_p = 0$$

by orthogonality

$$c_i u_i \cdot u_i = 0$$

$$c_i \| u_i \|^2 = 0$$

but since $u_i \neq 0$ then $c_i = 0$ which contradicts the assumption that $c_i \neq 0$. Thus the vectors are independent.

Matrix with orthonormal columns:

recall

Let $\{u_1, u_2, \ldots, u_p\} \subseteq \mathbb{R}^n$ then this is called an orthonormal set if $u_i \cdot u_j = \begin{cases} 0 & \text{whenever } i \neq j . \\ 1 & \text{whenever } i = j . \end{cases}$

Now put the u's in a matrix

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix} \in \mathbb{R}^{\overset{\text{rows}}{n} \times \overset{\text{columns.}}{p}} \qquad U^T = \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline \vdots \\ \hline u_p^T \end{bmatrix} \in \mathbb{R}^{p \times n}$$

$$U^T U = \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline \vdots \\ \hline u_p^T \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix} = I$$

In term of dot products, matrix-matrix multiplication is

$$\begin{array}{cc} & \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix} \\ \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline \vdots \\ \hline u_p^T \end{bmatrix} & \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots & u_1 \cdot u_p \\ u_2 \cdot u_1 & u_2 \cdot u_2 & \cdots & u_2 \cdot u_p \\ \vdots & & \ddots & \vdots \\ u_p \cdot u_1 & u_p \cdot u_2 & \cdots & u_p \cdot u_p \end{bmatrix} = U^T U = I \end{array}$$

So if the columns of U are orthonormal then $U^T U = I$, and if $U^T U = I$ then the columns of U are orthonormal.

Definition: an orthogonal matrix is a square matrix with orthonormal columns.

- Note $U^T U = I$ but since U is square this means $U^{-1} = U^T$

On you have inverses because the matrix is square then the transpose is the inverse... Conclusion.

$$UU^{-1} = I \quad \text{so} \quad \boxed{UU^T = I.}$$

only if U is square...

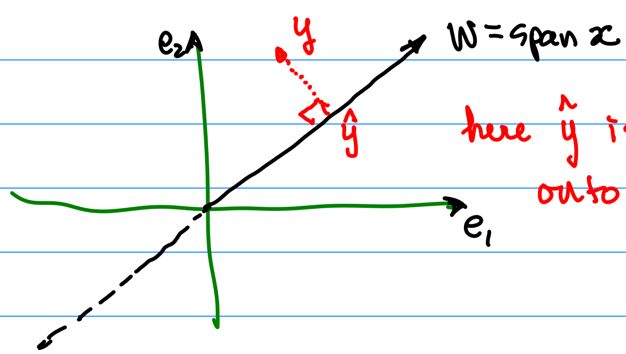
Projections onto higher dimensional subspaces...

$$f(y) = \hat{y} = \alpha x = \left(\frac{x \cdot y}{\|x\|^2}\right)x = \left(\frac{x}{\|x\|} \cdot y\right)\frac{x}{\|x\|} = (u \cdot y)u$$

$$= u(u \cdot y) = uu^T y = (uu^T)y$$

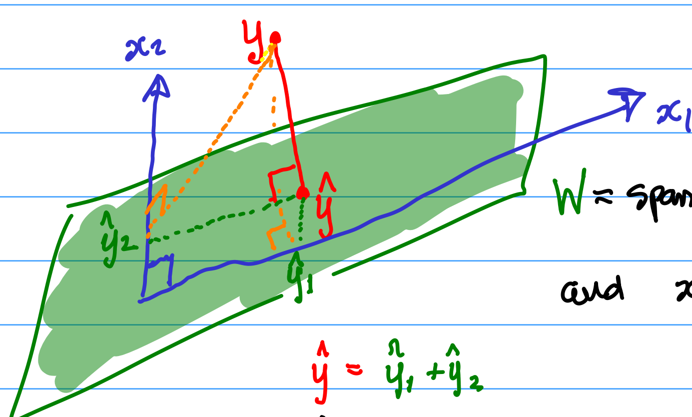
$$= Py$$

recall $u = \frac{x}{\|x\|}$



here $\hat{y}$ is the orthogonal projection of y onto the space W.

What does this look like when W has greater dimension?



$W = \text{span}\{x_1, x_2\}$

and $x_1 \cdot x_2 = 0$

$\hat{y} = \hat{y}_1 + \hat{y}_2$

to find $\hat{y}$ all I need is to find $\hat{y}_1$ and $\hat{y}_2$.

Here $\hat{y}_1$ is the projection of y in the x_1 direction

$\hat{y}_2$ is the projection of y in the x_2 direction

Let $u_1 = \frac{x_1}{\|x_1\|}$ be the unit vector in the x_1 direction

$u_2 = \frac{x_2}{\|x_2\|}$ be the unit vector in the x_2 direction

Then $W = \text{span}\{x_1, x_2\} = \text{span}\{u_1, u_2\}$

and

$$\hat{y}_1 = u_1 u_1^T y \qquad \hat{y}_2 = u_2 u_2^T y$$

from before

$= (uu^T)y$

Thus

$$\hat{y} = \hat{y}_1 + \hat{y}_2 = u_1 u_1^T y + u_2 u_2^T y$$

$$= \left[u_1 \mid u_2 \right] \left[\frac{u_1^T}{u_2^T} \right] y$$

Check

$$\left[u_1 \mid u_2 \right] \left[\frac{u_1^T}{u_2^T} \right] y = \left[u_1 \mid u_2 \right] \begin{bmatrix} u_1 \cdot y \\ u_2 \cdot y \end{bmatrix}$$

$$= u_1 (u_1 \cdot y) + u_2 (u_2 \cdot y) = u_1 u_1^T y + u_2 u_2^T y.$$

Define $U = \left[u_1 \mid u_2 \right]$ matrix with orthonormal columns that form a basis of W.

Then $\hat{y} = UU^T y$ this matrix projects y onto W.

works for any dimension subspace