- If $U \in \mathbb{R}^{m\times n}$ and U has orthonormal columns then $U^TU = I$

- If $U \in \mathbb{R}^{n\times n}$ and U has orthonormal columns then $U^TU = I$ but moreover $U^{-1} = U^T$ so $UU^T = I$

- If $U \in \mathbb{R}^{m\times n}$ what is UU^T in general?

  $P = UU^T$ the projection onto $W = \text{col}\, U$

Theorem 7 in Section 6.2

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$ ✓
b. $(U\mathbf{x})\cdot(U\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ ✓
c. $(U\mathbf{x})\cdot(U\mathbf{y}) = 0$ if and only if $\mathbf{x}\cdot\mathbf{y} = 0$

Note $U^TU = I$

(a) $\|x\| = \sqrt{x \cdot x}$ $\qquad \|x\|^2 = x \cdot x = x^Tx$

$\|Ux\| = \sqrt{Ux \cdot Ux}$ $\qquad \|Ux\|^2 = Ux \cdot Ux = (Ux)^T Ux = x^TU^TUx = x^Tx$

(b) $Ux \cdot Uy = (Ux)^T Uy = x^TU^TUy = x^Ty = x\cdot y$

The Orthogonal Decomposition Theorem

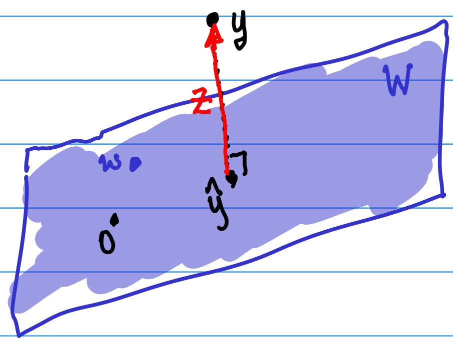
Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \qquad (1)$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any ortho~~gonal~~normal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \qquad (2)$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.



$$U = \left[u_1 \,\middle|\, u_2 \,\middle|\, \cdots \,\middle|\, u_p \right]$$

$$W = \operatorname{span}\{u_1, \dots, u_p\} = \operatorname{Col} U$$

$$\hat{y} = UU^T y \in \operatorname{Col} U = W$$

Let $z = y - \hat{y}$

then $\hat{y} + z = \hat{y} + y - \hat{y} = y$

So is $z \in W^{\perp}$? means $w \cdot z = 0$ for every $w \in W = \operatorname{Col} U$.

Since $w \in \operatorname{Col} U$ then there is x such that $w = Ux$.

$$w \cdot z = Ux \cdot (y - \hat{y}) = Ux \cdot (y - UU^T y) = Ux \cdot y - Ux \cdot UU^T y$$

$$= (Ux)^T y - (Ux)^T UU^T y = x^T U^T y - x^T \underbrace{U^T U}_{\text{identity}} U^T y$$

$$= x^T U^T y - x^T I \, U^T y = x^T U^T y - x^T U^T y = 0$$

Recall: $W \cap W^T = \{0\}$. Why:

$W = \text{col}\, U$

$W^T = \{v : v \cdot w = 0 \text{ for all } w \in W\}$

Suppose $v \in W^T \cap W$. Thus $v \in W^T$ and $v \in W$.

$v \in W^T$ means $v \cdot w = 0$ for all $w \in W$

$v \in W$ means we can take $w = v$ here

Therefore $v \cdot v = 0$ or $\|v\| = 0$.

Thus $v = 0$

Exam question: True / False

If $v \in W \cap W^T$ then $v = 0$.

Claim: The decomposition $y = \hat{y} + z$ where $\hat{y} \in W$ and $z \in W^\perp$ is unique. i.e. can only be done in one way.

Suppose there are two different decompositions

$y = \hat{y} + z$ and $y = \hat{y}_1 + z_1$

$\hat{y} \in W$, $z \in W^\perp$ $\qquad$ $\hat{y}_1 \in W$, $z_1 \in W^\perp$

Then

$$\hat{y} + z = \hat{y}_1 + z_1$$

$$\underbrace{\hat{y} - \hat{y}_1}_{\in W} = \underbrace{z_1 - z}_{\in W^\perp}$$

Thus $\hat{y} - \hat{y}_1 \in W \cap W^\perp$ so $\hat{y} = \hat{y}_1$ similarly $z = z_1$.

All of the above use an orthonormal basis for W. Thus we look for a way of making orthonormal basis.

$W = \text{span}\{b_1, b_2, \dots, b_p\}$ where $\{b_1, b_2, \dots, b_p\}$ is a basis

Let $A = \begin{bmatrix} b_1 | b_2 | \cdots | b_p \end{bmatrix}$ then $W = \text{col } A$.

Gram-Schmidt algorithm:

$z_1 = b_1$ $\qquad\qquad q_1 = \dfrac{z_1}{\|z_1\|}$

$z_2 = b_2 - (q_1 \cdot b_2) q_1 = b_1 - q_1 q_1^T b_1$ $\qquad\qquad q_2 = \dfrac{z_2}{\|z_2\|}$

$z_3 = b_3 - (q_1 \cdot b_3) q_1 - (q_2 \cdot b_3) q_2$ $\qquad\qquad q_3 = \dfrac{z_3}{\|z_3\|}$

$= b_3 - \begin{bmatrix} q_1 | q_1 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} b_3$

projection

$\vdots$

$z_p = b_p - (q_1 \cdot b_p) q_1 - \cdots - (q_{p-1} \cdot b_p) q_{p-1}$ $\qquad\qquad q_p = \dfrac{z_p}{\|z_p\|}$

Let $Q = \begin{bmatrix} q_1 | q_2 | \cdots | q_p \end{bmatrix}$