Example of finding the QR Factorization of a matrix using the Gram-Schmidt process...

$$A = \begin{bmatrix} b_1 | b_2 | \cdots | b_p \end{bmatrix} \qquad \boxed{Q = \begin{bmatrix} q_1 | q_2 | \cdots | q_p \end{bmatrix}} \qquad R = \begin{bmatrix} \|z_1\| & q_1 \cdot b_2 & q_1 \cdot b_3 & \cdots & q_1 \cdot b_p \\ 0 & \|z_2\| & q_2 \cdot b_3 & & q_2 \cdot b_p \\ 0 & 0 & \|z_3\| & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & q_{p-1} \cdot b_p \\ 0 & 0 & 0 & & \|z_p\| \\ 0 & 0 & 0 & \cdots & \end{bmatrix}$$

Algorithm:

$$z_1 = b_1 \qquad q_1 = \frac{z_1}{\|z_1\|}$$

$$z_2 = b_2 - (q_1 \cdot b_2) q_1 \qquad q_2 = \frac{z_2}{\|z_2\|}$$

$$\vdots$$

$$z_p = b_p - (q_1 \cdot b_p) q_1 - \cdots - (q_{p-1} \cdot b_p) q_{p-1} \qquad q_p = \frac{z_p}{\|z_p\|}$$

Example

$$\underset{4\times3}{A} = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \qquad \underset{4\times3}{Q} = \begin{bmatrix} -1/\sqrt{12} & 3/\sqrt{12} & -1/\sqrt{12} \\ 3/\sqrt{12} & 1/\sqrt{12} & -1/\sqrt{12} \\ 1/\sqrt{12} & 1/\sqrt{12} & 3/\sqrt{12} \\ 1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} \end{bmatrix} \qquad \underset{3\times3}{R} = \begin{bmatrix} \sqrt{12} & -36/\sqrt{12} & 6/\sqrt{12} \\ 0 & \sqrt{12} & 30/\sqrt{12} \\ 0 & 0 & \sqrt{12} \end{bmatrix}$$

$$z_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \qquad \|z_1\| = \sqrt{1+9+1+1} = \sqrt{12} \qquad q_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$(q_1 \cdot b_2) q_1$$

$$z_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \right) \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-36}{\sqrt{12}} \right) \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$-6 - 24 - 2 - 4 = -36$$

$$z_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad \|z_2\| = \sqrt{9+1+1+1} = \sqrt{12} \qquad q_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$z_3 = \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \left(\frac{1}{\sqrt{12}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix}\right)\frac{1}{\sqrt{12}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \left(\frac{1}{\sqrt{12}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix}\right)\frac{1}{\sqrt{12}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}$$

$$\frac{1}{\sqrt{12}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} = \frac{1}{\sqrt{12}}(-6+9+6-3) = \frac{6}{\sqrt{12}}$$

$$\frac{1}{\sqrt{12}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} = \frac{1}{\sqrt{12}}(18+3+6+3) = \frac{30}{\sqrt{12}}$$

$$z_3 = \begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \left(\frac{6}{\sqrt{12}}\right)\frac{1}{\sqrt{12}}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \left(\frac{30}{\sqrt{12}}\right)\frac{1}{\sqrt{12}}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix} = \frac{2}{2}\begin{bmatrix}6\\3\\6\\-3\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} - \frac{5}{2}\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}12+1-15\\6-3-5\\12-1-5\\-6-1+5\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-2\\-2\\6\\-2\end{bmatrix} = \begin{bmatrix}-1\\-1\\3\\-1\end{bmatrix}$$

$$\|z_3\| = \sqrt{1+1+9+1} = \sqrt{12}$$

$$q_3 = \frac{1}{\sqrt{12}}\begin{bmatrix}-1\\-1\\3\\-1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{12}\\-1/\sqrt{12}\\3/\sqrt{12}\\-1/\sqrt{12}\end{bmatrix}$$

Therefore the $A=QR$ factorization is:

$$\begin{bmatrix}-1 & 6 & 6\\3 & -8 & 3\\1 & -2 & 6\\1 & -4 & -3\end{bmatrix} = \begin{bmatrix}-1/\sqrt{12} & 3/\sqrt{12} & -1/\sqrt{12}\\3/\sqrt{12} & 1/\sqrt{12} & -1/\sqrt{12}\\1/\sqrt{12} & 1/\sqrt{12} & 3/\sqrt{12}\\1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12}\end{bmatrix}\begin{bmatrix}\sqrt{12} & -36/\sqrt{12} & 6/\sqrt{12}\\0 & \sqrt{12} & 30/\sqrt{12}\\0 & 0 & \sqrt{12}\end{bmatrix}$$

matrix with orthonormal columns

upper triangular matrix

$Q^TQ = I$

QQ^T is projection onto Col A.

Chapter 7: Eigenvalues and Eigenvectors and Orthogonality.

Symmetric matrix A means $A^T = A \in \boxed{\mathbb{R}^{n\times n}}$ note A is a real matrix.

recall that $Ax \cdot y = x \cdot A^T y$ (why transpose is useful)

Symmetric means $Ax \cdot y = x \cdot Ay$.

Properties of symmetric matrices:

① Eigenvalues are real. Why?

Checked this using the quadratic formula for 2x2 matrices a couple weeks ago..., where in the notes?

Suppose λ is an eigenvalue of A
and v is an corresponding eigenvector of A

Therefore $Av = \lambda v$. Claim is $A = A^T$ implies λ is real
or equivalently that $\lambda = \bar{\lambda}$.

Note: if $\lambda \neq \bar{\lambda}$ then v is not a real vector either.

$\|v\| \neq 0$ since v is an eigenvector.

$$\sqrt{|v_1|^2 + |v_2|^2 + \cdots |v_n|^2} = \sqrt{v_1\bar{v}_1 + v_2\bar{v}_2 + \cdots + v_n\bar{v}_n} = \sqrt{v \cdot \bar{v}}$$

Compute

$$\lambda v \cdot \bar{v} = Av \cdot \bar{v} = v \cdot A\bar{v} = v \cdot \bar{A}\bar{v} = v \cdot \overline{Av} = v \cdot \overline{\lambda v}$$

$$= v \cdot \bar{\lambda}\bar{v} = \bar{\lambda} v \cdot \bar{v}$$

Thus $(\lambda - \bar{\lambda}) v \cdot \bar{v} = 0$ since $v \neq 0$ then $\lambda - \bar{\lambda} = 0$.