Given $A \in \mathbb{R}^{m\times n}$ let $B = \underset{n\times m}{A^T} \underset{m\times n}{A}$ $\in \mathbb{R}^{n\times n}$ and apply the spectral theorem to $B$ to obtain an orthonormal basis of eigenvectors such that $u_i \in \mathbb{R}^n$

$$\underbrace{Bu_i = \lambda_i u_i}_{\text{eigenvectors}} \qquad \text{and} \qquad \underbrace{u_i \cdot u_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}}_{\text{orthonormal.}}$$

Relate the eigenvectors back to $A$:

$$y_i = \underset{m\times n}{A}\,\underset{n}{u_i} \in \mathbb{R}^m$$

$$\|y_i\| = \sqrt{y_i \cdot y_i} = \sqrt{Au_i \cdot Au_i} = \sqrt{u_i^T A^T A u_i} = \sqrt{u_i^T B u_i}$$

$$= \sqrt{u_i^T \lambda_i u_i} = \sqrt{\lambda_i}\ u_i \cdot u_i = \sqrt{\lambda_i}$$

Also,

$$y_i \cdot y_j = Au_i \cdot Au_j = u_i^T B u_j = \lambda_j u_i^T u_j = \begin{cases} \lambda_i & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

Therefore

$$z_i = \frac{y_i}{\|y_i\|} = \frac{y_i}{\sqrt{\lambda_i}} \in \mathbb{R}^m$$ are orthonormal vectors and if $m=n$ they form an orthonormal basis

Define $U = [u_1 | u_2 | \cdots | u_n] \in \mathbb{R}^{n\times n}$ $\qquad V = [z_1 | z_2 | \cdots | z_n] \in \mathbb{R}^{m\times n}$

Now

$$AU = A[u_1 | u_2 | \cdots | u_n] = [Au_1 | Au_2 | \cdots | Au_n] = [y_1 | y_2 | \cdots | y_n]$$

$$= [\sqrt{\lambda_1} z_1 | \sqrt{\lambda_2} z_2 | \cdots | \sqrt{\lambda_n} z_n] = [z_1 | z_2 | \cdots | z_n] \underbrace{\begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \ \sqrt{\lambda_n} \end{bmatrix}}_{\Sigma} = V\Sigma$$

Thus $AU = V\Sigma$ since $U$ is square then $UU^T = I$

$$AUU^T = V\Sigma U^T$$

$$A = V\Sigma U^T \leftarrow \text{singular value decomposition}$$

Note that when making this factorization, it is standard to order the eigenvalues so $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n$.

Example: $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ Find the singular value decomposition of $A$.

$$B = A^T A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

Find an orthonormal basis of eigenvectors of $B$.

$$\chi_B(\lambda) = \det\begin{pmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = (8-\lambda)(5-\lambda) - 4 = \lambda^2 - 13\lambda + 40 - 4$$

$$= \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$$

Therefore $\lambda_1 = 9$ and $\lambda_2 = 4$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Find the eigenvectors...

$\lambda = 9$

$$\begin{bmatrix} 8-9 & 2 \\ 2 & 5-9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

$-x_1 + 2x_2 = 0$

$x_1 = 2x_2$

so $u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 4$

$$\begin{bmatrix} 8-4 & 2 \\ 2 & 5-4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$2x_1 + x_2 = 0$

$x_2 = -2x_1$

$u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$U = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$AU = V\Sigma$

$V = AU\Sigma^{-1}$

what is $V$?  $A = V \Sigma U^T$

$V = [z_1 | z_2]$ where $z_1 = \frac{Au_1}{\sqrt{\lambda_1}} = \frac{Au_1}{\|Au_1\|}$  $z_2 = \frac{Au_2}{\sqrt{\lambda_2}} = \frac{Au_2}{\|Au_2\|}$

$$Au_1 = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{3}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\frac{Au_1}{\|Au_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Au_2 = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \frac{2}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\frac{Au_2}{\|Au_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \qquad U^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

Therefore $A = V \Sigma U^T$ is

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & -2\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}3 & 4\\ 6 & -2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & -2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}10 & -5\\ 10 & 10\end{bmatrix} = \begin{bmatrix}2 & -1\\ 2 & 2\end{bmatrix}$$

ok ☑

**13.** Let $B = A^T A$ where $A$ is given by

$$A = \begin{bmatrix} 2 & -1 \\ -2/5 & 11/5 \end{bmatrix}.$$

Note that $B$ has eigenvalues and eigenvectors given by

$$\lambda_1 = 8, \quad x_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \lambda_2 = 2, \quad x_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

$\|x_1\| = \sqrt{9+16} = 5$ $\qquad$ $\|x_2\| = 5$

Find the singular value decomposition $A = V\Sigma U^T$ where $\Sigma$ is a diagonal matrix and $U$ and $V$ are orthogonal matrices.

$$\Sigma = \begin{bmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\end{bmatrix} = \begin{bmatrix}\sqrt{8} & 0\\ 0 & \sqrt{2}\end{bmatrix}$$

$$U = \begin{bmatrix} u_1 \mid u_2 \end{bmatrix} = \begin{bmatrix}-3/5 & 4/5\\ 4/5 & 3/5\end{bmatrix}$$

$$y_1 = \begin{bmatrix} 2 & -1 \\ -2/5 & 11/5 \end{bmatrix}\begin{bmatrix}-3/5\\ 4/5\end{bmatrix} = \frac{1}{25}\begin{bmatrix}10 & -5\\ -2 & 11\end{bmatrix}\begin{bmatrix}-3\\ 4\end{bmatrix} = \frac{1}{25}\begin{bmatrix}-30 & -20\\ & 6+44\end{bmatrix} = \frac{1}{25}\begin{bmatrix}-50\\ 50\end{bmatrix}$$

$$z_1 = \frac{y_1}{\|y_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}$$

$$V = \begin{bmatrix}-1/\sqrt{2} & \\ 1/\sqrt{2} & \end{bmatrix} z_2$$

didn't have time but the same idea

$y_2 = Au_2 \qquad z_2 = \frac{y_2}{\|y_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$