ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1] $r_i \leftarrow r_i + \alpha r_j \quad i \neq j$ add α times row j to row i
2. (Interchange) Interchange two rows. $r_i \leftrightarrow r_j \quad i \neq j$ switch row i with row j
3. (Scaling) Multiply all entries in a row by a nonzero constant. $r_i \leftarrow \alpha r_i \quad \alpha \neq 0$ rescale row i by α.

Interchange: $r_1 \leftrightarrow r_2$

$$g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0x_1 + 1x_2 + 0x_3 \\ 1x_1 + 0x_2 + 0x_3 \\ 0x_1 + 0x_2 + 1x_3 \end{bmatrix}$$

$$g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Scaling

$r_2 \leftarrow 5r_2$

$$g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 5x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

replacement (elimination)

$r_3 \leftarrow r_3 - 2r_1$

$$g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ -2x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Identity (do nothing)

$$g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note to find the matrix corresponding to the row operation perform the row operation on the identity matrix.

$$I = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$$

$r_3 \leftarrow r_3 - 2r_1$

$\times(-2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

← the matrix the represents the operation $r_3 \leftarrow r_3 - 2r_1$

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13. $x_1 - 3x_3 = 8$
$2x_1 + 2x_2 + 9x_3 = 7$
$x_2 + 5x_3 = -2$

← Consider solving this system...

write as augmented matrix.

Now do row operations.

$$[A|b] = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \end{array}$$

$r_2 \leftarrow r_2 - 2r_1$

the pivot

Goal do row operations until these entries are all zero

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

$r_3 \leftarrow r_3 - \frac{1}{2} r_2$

new pivot

remark: this operation will lead to fractions, but it will work...

idea: rearrange the equations to make the arithmetic easier...

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

$r_2 \leftrightarrow r_3$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix}$$

different pivot

$r_3 \leftarrow r_3 - 2r_2$

$$\begin{array}{r} 15 \\ -10 \\ \hline 5 \end{array} \qquad \begin{array}{r} -9 \\ +4 \\ \hline -5 \end{array}$$

$$
\begin{array}{cccc}
x_1 & x_2 & x_3 & b \\
\end{array}
$$

$$
\left[\begin{array}{ccc:c}
1 & 0 & -3 & 8 \\
0 & 1 & 5 & -2 \\
0 & 0 & 5 & -5
\end{array}\right]
\qquad
\begin{aligned}
x_1 \quad - 3x_3 &= 8 \\
x_2 + 5x_3 &= -2 \\
5x_3 &= -5
\end{aligned}
$$

A system of equations corresponding to when there is a triangle of zeros in the lower corner can be solved by back substitution.

because you start with the last variable

$$
\begin{aligned}
x_1 \quad - 3x_3 &= 8 \\
x_2 + 5x_3 &= -2 \\
5x_3 &= -5
\end{aligned}
\qquad
\begin{cases}
x_1 = 8 + 3x_3 = 5 \\
x_2 = -2 - 5x_3 = -2 + 5 = 3 \\
x_3 = -1
\end{cases}
$$

Answer

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}
$$

<u>Remark:</u> When doing elimination by hand we pick small but not zero pivots to avoid fractions with large denominators.

When a computer picks the pivot (by swapping rows) it chooses the largest (in magnitude) so dividing by the large pivots reduces rounding error.

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

✓ **1.** All nonzero rows are above any rows of all zeros.

✓ **2.** Each leading entry of a row is in a column to the right of the leading entry of the row above it.

✓ **3.** All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

? **4.** The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right] \qquad r_3 \leftarrow \frac{1}{5} r_3$$

not a 1

not 0

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \begin{array}{l} r_1 \leftarrow r_1 + 3r_3 \\ r_2 \leftarrow r_1 - 5r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$