Another example:

$$x_1 - 7x_2 + 6x_4 = 5$$
$$x_3 - 2x_4 = -3$$
$$-x_1 + 7x_2 - 4x_3 + 2x_4 = 8$$

$$A = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ -3 \\ 8 \end{bmatrix}$$

Augmented matrix

pivot

$$[A \vdots b] = \left[\begin{array}{cccc:c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 8 \end{array}\right] \qquad r_3 \leftarrow r_3 + r_1$$

next

$$\left[\begin{array}{cccc:c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 13 \end{array}\right] \qquad r_3 \leftarrow r_3 + 4r_2$$

$$\left[\begin{array}{cccc:c} \boxed{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \boxed{1} & -2 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

A

alg notation

wasn't a pivot of A because it is in the b column.

$$x_1 - 7x_2 + 6x_4 = 5$$
$$x_3 - 2x_4 = -3$$
$$0 = 1$$

contradiction so no solutions

pg 39 section 1.4 Theorem 4

Theorem about when $Ax = b$ has a solution.

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

✓ a. For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of A. ?

c. The columns of A span $\mathbb{R}^m$. ?

✓ d. A has a pivot position in every row.

we see from using the echelon form that if every row has a pivot then there is a solution... That's actually how we find the solution!

12. $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$[A|b] = \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right] \qquad r_2 \leftarrow r_2 + 3r_1$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{5} & 5 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right] \qquad r_3 \leftarrow r_3 - r_2$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{5} & 5 & 1 \\ 0 & 0 & \boxed{-2} & -2 \end{array}\right]$$

triangle of zeros means we have the echelon form

there is a pivot in every row of A.

means... For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

$$\text{span}\{v_1, v_2, \ldots, v_p\} = \{c_1v_1 + c_2v_2 + \cdots + c_pv_p \;:\; c_1, c_2, \ldots, c_p \in \mathbb{R}\}$$

Column representation of matrix-vector product

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \qquad Ax = v_1x_1 + v_2x_2 + \cdots + v_px_p$$

means c_1, c_2 and c_3 are real numbers that is, scalars.

$$\text{span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\0\\0\end{bmatrix}\right\} = \left\{c_1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + c_2\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}-1\\2\\0\\0\end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R}\right\}$$

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \end{array}$$

What linear function does A correspond to?

$$f(x) = Ax = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + 0x_2 + 2x_3 \\ 3x_1 + x_2 + 0x_3 \\ 4x_1 + 0x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}x_1 + \begin{bmatrix}1\\0\\1\\0\end{bmatrix}x_2 + \begin{bmatrix}-1\\2\\0\\0\end{bmatrix}x_3$$

$$\text{span}\{v_1, \ldots, v_p\} = \{Ax : x \in \mathbb{R}^p\} \quad \text{where } A = \begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix}$$

$$= \text{range } f = \boxed{\text{Col } A}$$

So there's lots of different notation that all means the same thing

$$f(x) = Ax = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + 0x_2 + 2x_3 \\ 3x_1 + x_2 + 0x_3 \\ 4x_1 + 0x_2 + 0x_3 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \\ 4 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

dot products

$$= \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + 0x_2 + 2x_3 \\ 3x_1 + x_2 + 0x_3 \\ 4x_1 + 0x_2 + 0x_3 \end{bmatrix} \Bigg| = \begin{bmatrix} 1\cdot 5 + 1\cdot 6 + (-1)7 \\ 2\cdot 5 + 0\cdot 6 + 2\cdot 7 \\ 3\cdot 5 + 1\cdot 6 + 0\cdot 7 \\ 4\cdot 5 + 0\cdot 6 + 0\cdot 7 \end{bmatrix}$$

between the rows of the matrix and x.

$(x_1, x_2, x_3) = (5, 6, 7)$

In general if A has m rows like

$$A = \begin{bmatrix} a_1 \\ \hdashline a_2 \\ \hdashline \vdots \\ \hdashline a_m \end{bmatrix}$$

then

$$Ax = \begin{bmatrix} a_1 \bullet x \\ a_2 \bullet x \\ \vdots \\ a_m \bullet x \end{bmatrix}$$

dot products of the vector a_i with x.

row representation of matrix-vector products.

Column representation of matrix-vector product

$$Ax = v_1 x_1 + v_2 x_2 + \cdots + v_p x_p$$

Definition of the Null space of a matrix $A \in \mathbb{R}^{m \times n}$

rows columns

means A has m rows and n columns

$$\text{Nul } A = \{ x \in \mathbb{R}^n : Ax = 0 \}.$$

- Connect this to the free variables next time...