Think of an $m \times n$ matrix $A \in \mathbb{R}^{\overset{\text{rows, equations}}{m} \times \overset{\text{columns, variables}}{n}}$ as a linear function $f(x) = Ax$

$$\begin{array}{cc} & \text{4 columns} \\ \text{3 rows} & A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \in \mathbb{R}^{3 \times 4} \end{array}$$

And $f : \underset{\text{vectors with 4 components}}{\mathbb{R}^4} \longrightarrow \mathbb{R}^3$ ← range is a vector with 3 components

$$\operatorname{Col} A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m \leftarrow \text{number of rows}$$

$$\operatorname{Nul} A = \{ x \in \mathbb{R}^n : Ax = 0 \}$$

Theorem 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

5.
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

Solve this system is the same as finding Nul A

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$$

Solve $Ax = 0$.

I already know $x = 0$ is one solution. Are there any others?

Solve $Ax = 0$.

Make the echelon / reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \qquad r_2 \leftarrow r_2 + 4r_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \qquad \begin{array}{l} r_3 \leftarrow r_3 + r_2 \\ r_1 \leftarrow r_1 - r_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad r_2 \leftarrow \frac{1}{3} r_2$$

$$\begin{array}{cccc} P & P & F & \\ x_1 & x_2 & x_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \end{array} \qquad \begin{array}{l} x_1 - 5x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}$$

Solve pivot variables in terms of the free variables

$$x_1 = 5x_3$$
$$x_2 = -2x_3$$

solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$\boxed{\operatorname{Nul} A = \{ x \in \mathbb{R}^n : Ax = \vec{0}_m \}}$$

$$= \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} x_3 : x_3 \in \mathbb{R} \right\} = \left\{ \underbrace{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}_{\text{matrix}} \underbrace{\begin{bmatrix} x_3 \end{bmatrix}}_{\text{vector}} : x_3 \in \mathbb{R} \right\} = \operatorname{Col}\left(\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right)$$

19. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \quad x_4 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \quad x_4 \\ -3x_2 - 6x_3 &= -3 \quad x_4 \end{aligned}$$

$x_4 = 1$

5.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

Make the echelon/reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right] \qquad r_2 \leftarrow r_2 + 4r_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right] \qquad \begin{aligned} r_3 &\leftarrow r_3 + r_2 \\ r_1 &\leftarrow r_1 - r_2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad r_2 \leftarrow \frac{1}{3} r_2$$

← lucky, otherwise there'd be no solutions

$$\begin{array}{cccc} P & P & F & \\ x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 5x_3 &= -2 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

Solve pivot variables in terms of the free variables

$$x_1 = -2 + 5x_3$$

$$x_2 = 1 - 2x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2+5x_3 \\ 1-2x_3 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\text{Shift}} + \underbrace{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}_{\text{Nul } A} x_3$$

is actually one of the solutions to

$Ax = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

So the theorem explains what happens when we solve the problem as in the previous section. The idea is to explain a general understanding of what's going on. That is what theory means (in mathematical contexts).