Online homework is due tonight and there is more for later so you can work ahead. New assignments will open over time so please keep checking...

1.7 Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$\overbrace{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p}^{\text{span}} = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p}_{\text{span}\{v_1, \dots, v_p\}} = \mathbf{0} \qquad (2)$$

Realize linear combinations of vectors using Matrix-vector mult.

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \in \mathbb{R}^{n \times p}$$

n rows

p column

The only solution to $Ax=0$ is $x=0$

means

The vectors v_i are independent

linear dependence is the oposite of independent

There exists a non-zero vector $c \in \mathbb{R}^p$ such that $Ac=0$.

means

The vectors v_i are dependent

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Is the set $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ linearly independent or dependent?

Note $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not linear combination of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for every } c_1 \in \mathbb{R}$$

On the other hand

$$1 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{when } c_2 = 0.$$

Echelon form of A

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \qquad r_2 \leftarrow r_2 - 2r_1$$

$$\begin{array}{cc} F & P \\ \end{array}$$
$$\begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

one pivot

Echelon form of A ... actually the reduced echelon form by luck.

If there are free variables then the equation $Ax = 0$ has more than one solution. Take c to be one of the non-zero solutions. Conclude the columns of A were dependent.

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \in \mathbb{R}^{\overset{\text{rows}}{n} \times \overset{\text{columns}}{p}}$$

p columns

n rows since $v_i \in \mathbb{R}^n$

$p > n$ means more columns than rows...

more variables than equations

Thus there are free variables

Think about solving $Ax = 0$. Perform row operations until A is in echelon form.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \in \mathbb{R}^{3\times 4}$$

$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \end{bmatrix}$$

generally what happens...

$$\begin{array}{cccc} P & P & P & F \end{array}$$
$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \end{bmatrix}$$

maybe that was zero by luck...

$$\begin{array}{cccc} P & F & P & P \end{array}$$
$$\begin{bmatrix} * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

even is the special cases there are at most 3 pivots, so always at least 1 free variable...

this one might be zero then only 2 pivot variables...

You can have no more pivots than the number of rows, and since there are more columns than rows there must be a column without a pivot.

Review of Matrix vector multiplication $A \in \mathbb{R}^{m \times n}$

row representation

$$Ax = \begin{bmatrix} \text{---} \; a_1 \; \text{---} \\ \text{---} \; a_2 \; \text{---} \\ \vdots \\ \text{---} \; a_m \; \text{---} \end{bmatrix} x = \begin{bmatrix} a_1 \cdot x \\ a_2 \cdot x \\ \vdots \\ a_m \cdot x \end{bmatrix}$$

column representation

$$Ax = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} x = v_1 x_1 + v_2 x_2 + \cdots + v_n x_n$$

Example:

$$Ax = \begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (4 \;\; -3 \;\; 1) \cdot (-3 \;\; -1 \;\; 2) \\ (5 \;\; -2 \;\; 5) \cdot (-3 \;\; -1 \;\; 2) \\ (-6 \;\; 2 \;\; -3) \cdot (-3 \;\; -1 \;\; 2) \end{bmatrix}$$

$$= \begin{bmatrix} -12 + 6 + 2 \\ -15 + 2 + 10 \\ 18 - 2 - 6 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}.$$

$$Ax = \begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}(-3) + \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}(-1) + \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} 2$$

$$= \begin{bmatrix} -12 \\ -5 \\ 18 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 10 \\ -6 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}.$$

Linear Transformations

A function $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is called a linear transformation if

- $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^m$
- $T(cu) = cT(u)$ for all $c \in \mathbb{R}$ and $u \in \mathbb{R}^m$.

Claim there exists a matrix $A \in \mathbb{R}^{n \times m}$ such that $T(x) = Ax$

usually consider $m \times n$ here...

more like before

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Standard basis of $\mathbb{R}^n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Note $e_i \in \mathbb{R}^n$ with all all zero entries except the ith entry which is 1.

Think about $\mathbb{R}^3$. $x \in \mathbb{R}$. factor out the entries of x

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$$

Explain the claim next time