2 Matrix Algebra

rows columns

$$A=\begin{bmatrix}2&3&5\\1&0&7\end{bmatrix}\in\mathbb{R}^{2\times 3}\qquad B=\begin{bmatrix}1&0&-2\\5&2&1\end{bmatrix}$$

Recall: The columns are related to the variables x_1, x_2, x_3 which are inputs to the function.

$$A+B=\begin{bmatrix}2&3&5\\1&0&7\end{bmatrix}+\begin{bmatrix}1&0&-2\\5&2&1\end{bmatrix}=\begin{bmatrix}3&3&3\\6&2&8\end{bmatrix}$$

$$3\begin{bmatrix}2&3&5\\1&0&7\end{bmatrix}=\begin{bmatrix}6&9&15\\3&0&21\end{bmatrix}$$

$AB = ?$ whats going on

$f:\mathbb{R}^3\to\mathbb{R}^2$

$$f(x)=Ax=\begin{bmatrix}2&3&5\\1&0&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2x_1+3x_2+5x_3\\x_1+0x_2+7x_3\end{bmatrix}\in\mathbb{R}^2$$

$g:\mathbb{R}^3\to\mathbb{R}^2$

$$g(x)=Bx=\begin{bmatrix}1&0&-2\\5&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1+0x_2-2x_3\\5x_1+2x_2+x_3\end{bmatrix}$$

whats this?

$$(f\cdot g)(x)=f(x)\cdot g(x)=\begin{bmatrix}2x_1+3x_2+5x_3\\x_1+0x_2+7x_3\end{bmatrix}\cdot\begin{bmatrix}x_1+0x_2-2x_3\\5x_1+2x_2+x_3\end{bmatrix}$$

doesn't make sense?
but not a linear function anymore.
also not the same as the element by element product of A with B...

composition of functions

$$(f\circ g)(x)=f(g(x))$$

also doesn't make sense because output of g is in $\mathbb{R}^2$ but f need something in $\mathbb{R}^3$

$f: \mathbb{R}^3 \to \mathbb{R}^2,$ $\qquad f(x) = Ax \qquad A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 7 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

$h: \mathbb{R}^2 \to \mathbb{R}^3,$ $\qquad h(x) = Cx \qquad C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$

Composition.

$$\mathbb{R}^2 \xrightarrow{h} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$$

$f \circ h : \mathbb{R}^2 \to \mathbb{R}^2$

$$(f \circ h)(x) = f(h(x)) = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} h(x) \end{bmatrix} = \begin{bmatrix} (2,3,5) \cdot h(x) \\ (1,0,7) \cdot h(x) \end{bmatrix}$$

$$h(x) = Cx = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} x_2$$

$$(2,3,5) \cdot h(x) = (2,3,5) \cdot \left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} x_2 \right)$$

$$= (2,3,5) \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} x_1 + (2,3,5) \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} x_2$$

$$(1,0,7) \cdot h(x) = (1,0,7) \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} x_1 + (1,0,7) \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} x_2$$

$$\begin{bmatrix} (2,3,5)\cdot h(x) \\ (1,0,7)\cdot h(x) \end{bmatrix} = \underbrace{\begin{bmatrix} (2,3,5)\cdot \begin{bmatrix}1\\0\\-2\end{bmatrix} x_1 + (2,3,5)\cdot \begin{bmatrix}3\\1\\-1\end{bmatrix} x_2 \\ (1,0,7)\cdot \begin{bmatrix}1\\0\\-2\end{bmatrix} x_1 + (1,0,7)\cdot \begin{bmatrix}3\\1\\-1\end{bmatrix} x_2 \end{bmatrix}}$$

linear function - therefore it can be written as a matrix-vector multiplication

$$(f \circ h)(x) = \begin{bmatrix} (2,3,5)\cdot \begin{bmatrix}1\\0\\-2\end{bmatrix} & (2,3,5)\cdot \begin{bmatrix}3\\1\\-1\end{bmatrix} \\ (1,0,7)\cdot \begin{bmatrix}1\\0\\-2\end{bmatrix} & (1,0,7)\cdot \begin{bmatrix}3\\1\\-1\end{bmatrix} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$C \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \right.$$

$$A \left\{ \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 7 \end{bmatrix} \right. \underbrace{\begin{bmatrix} -8 & 4 \\ -13 & -4 \end{bmatrix}}_{AC}$$

$A \in \mathbb{R}^{2\times 3}$ $\qquad$ $C \in \mathbb{R}^{3\times 2}$

$$\underset{2\times\cancel{3}}{A}\ \underset{\cancel{3}\times 2}{C} = \underset{2\times 2}{(AC)}$$

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 3}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \in \mathbb{R}^{2\times 3},$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} \mathbb{R}^{2\times 2}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \mathbb{R}^{2\times 1}$$

1. $-2A$, $B - 2A$, $\underset{2\times 3\ \ 2\times 2}{AC}$, $\underset{2\times 2\ \ 2\times 2}{CD}$

   don't match

2. $A + 2B$, $3C - E$, CB, EB

$$-2A = -2\begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 6 & -4 \end{bmatrix}$$

AC is undefined because the A has 3 columns but C has only 2 rows so they don't match.

$$CD = \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$$

The Transpose of a Matrix

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

swaps rows with columns of a matrix

EXAMPLE 8 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$ ■